Quantum Complexity

A Brief Overview

Fernando Granha Jeronimo

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How powerful are quantum computers?

Are there advantages of quantum computers over classical ones?

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Are there limitations to quantum computers?

Complexity theory sheds light on these questions!

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Such proof seems beyond our current techniques!

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In the black-box model, we can prove lower bounds!

We will see our first lower bound (or impossibility) result

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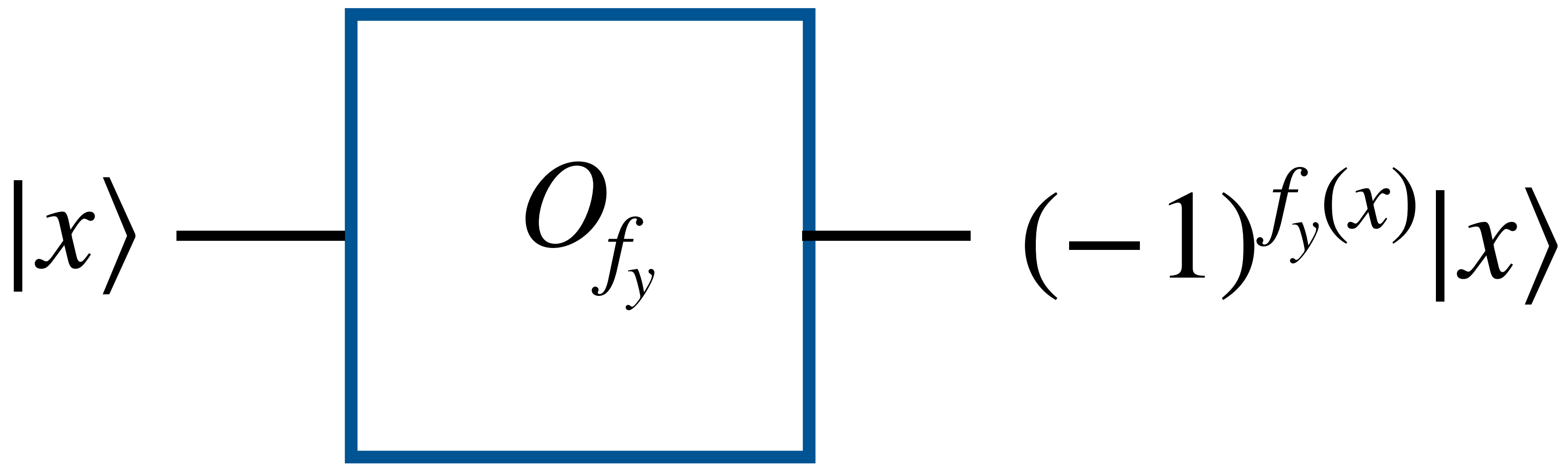
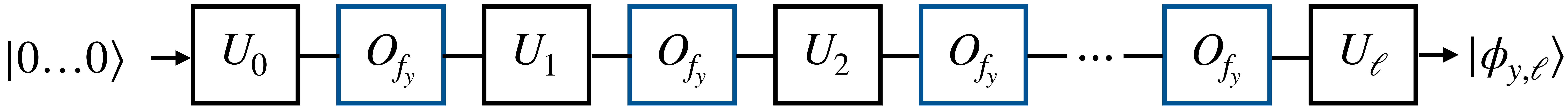


$$O_{f_y}|x\rangle = (-1)^{f_y(x)}|x\rangle$$

Quantum Query Algorithm

An arbitrary ℓ-query quantum algorithm can be expressed as



for some fixed choice of unitaries $U_0, U_1, \ldots, U_\ell$

Hybrid Method

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Break the analysis into a sequence of slightly different computations

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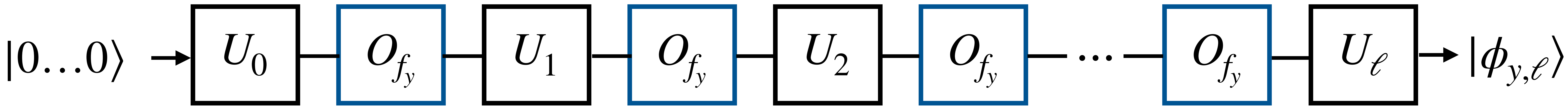

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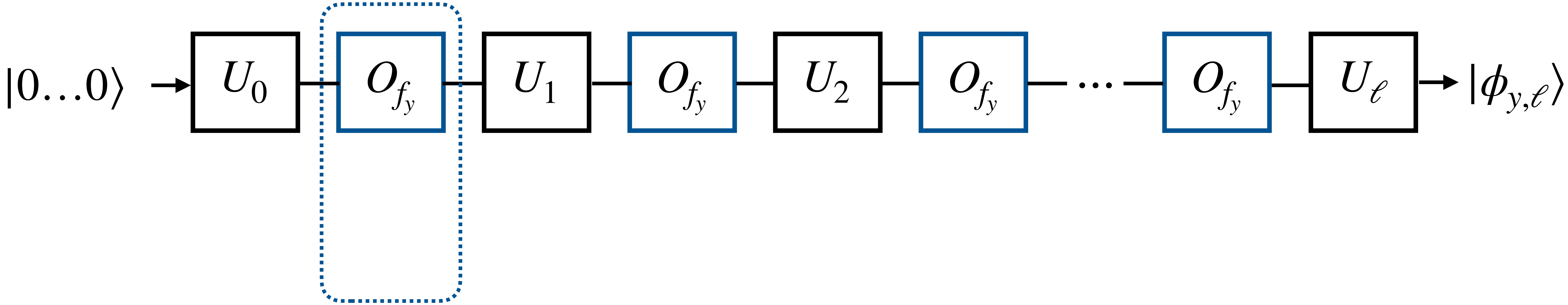

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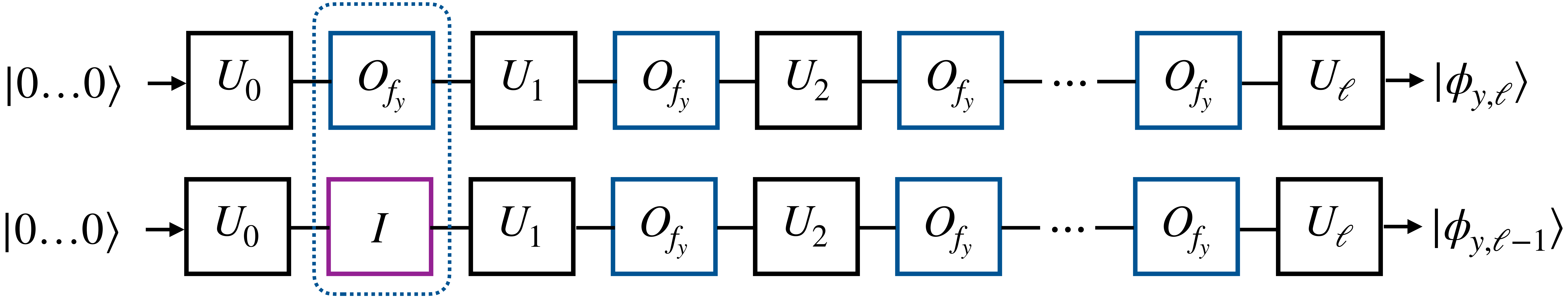

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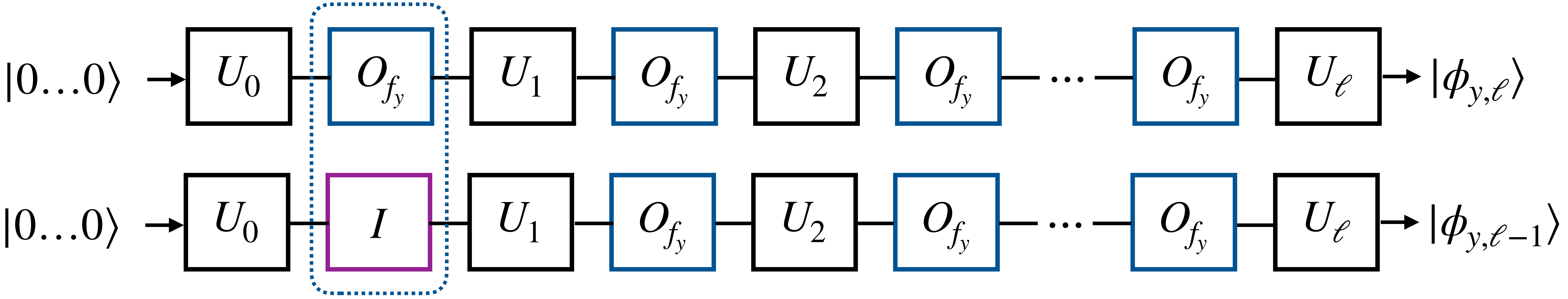

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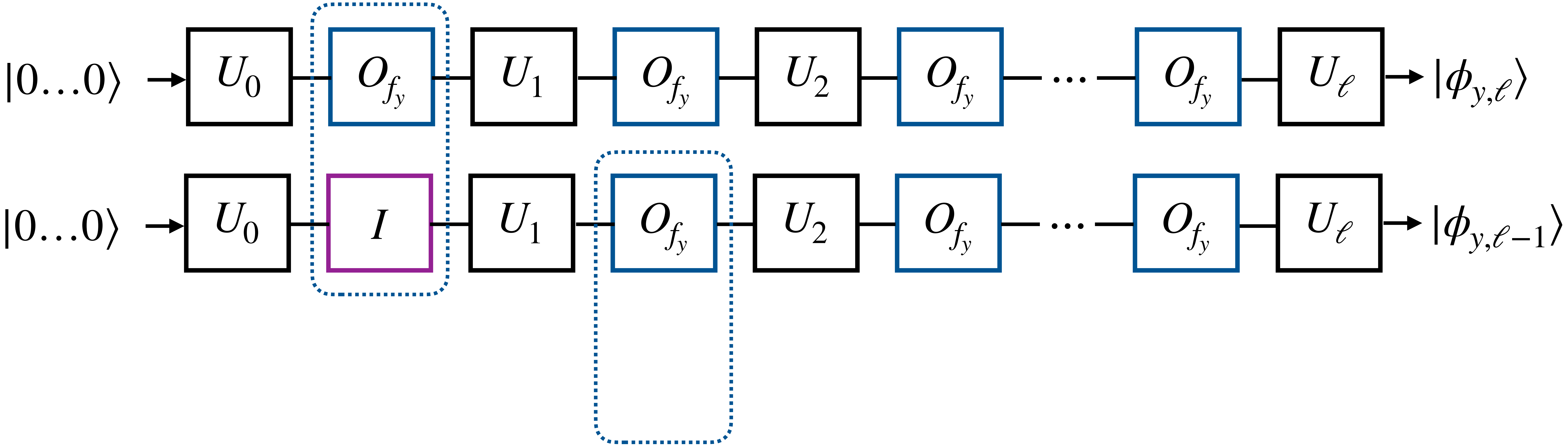

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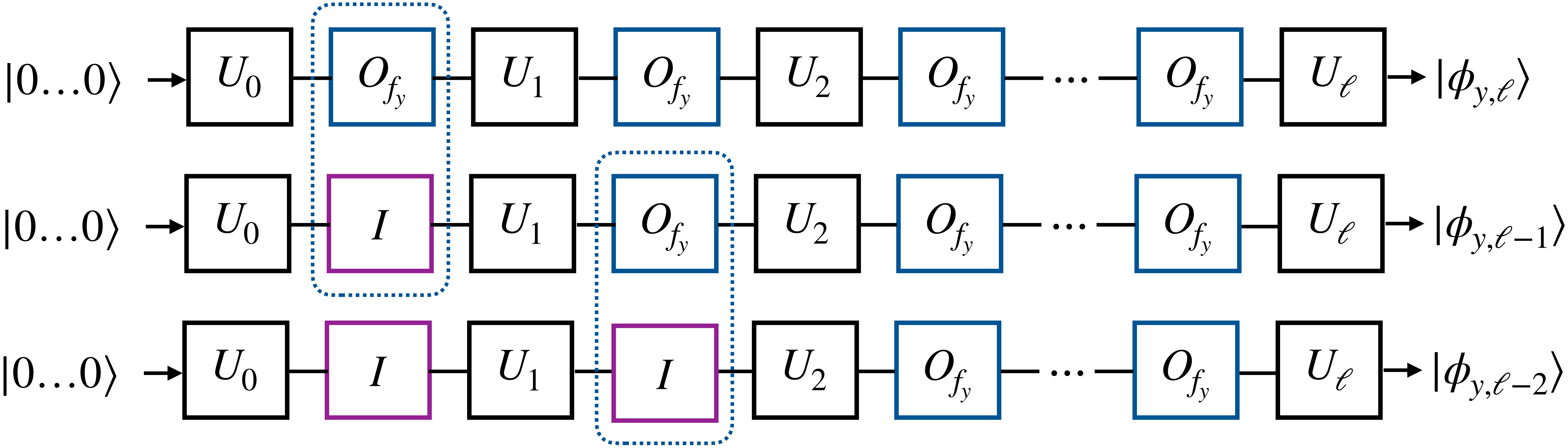

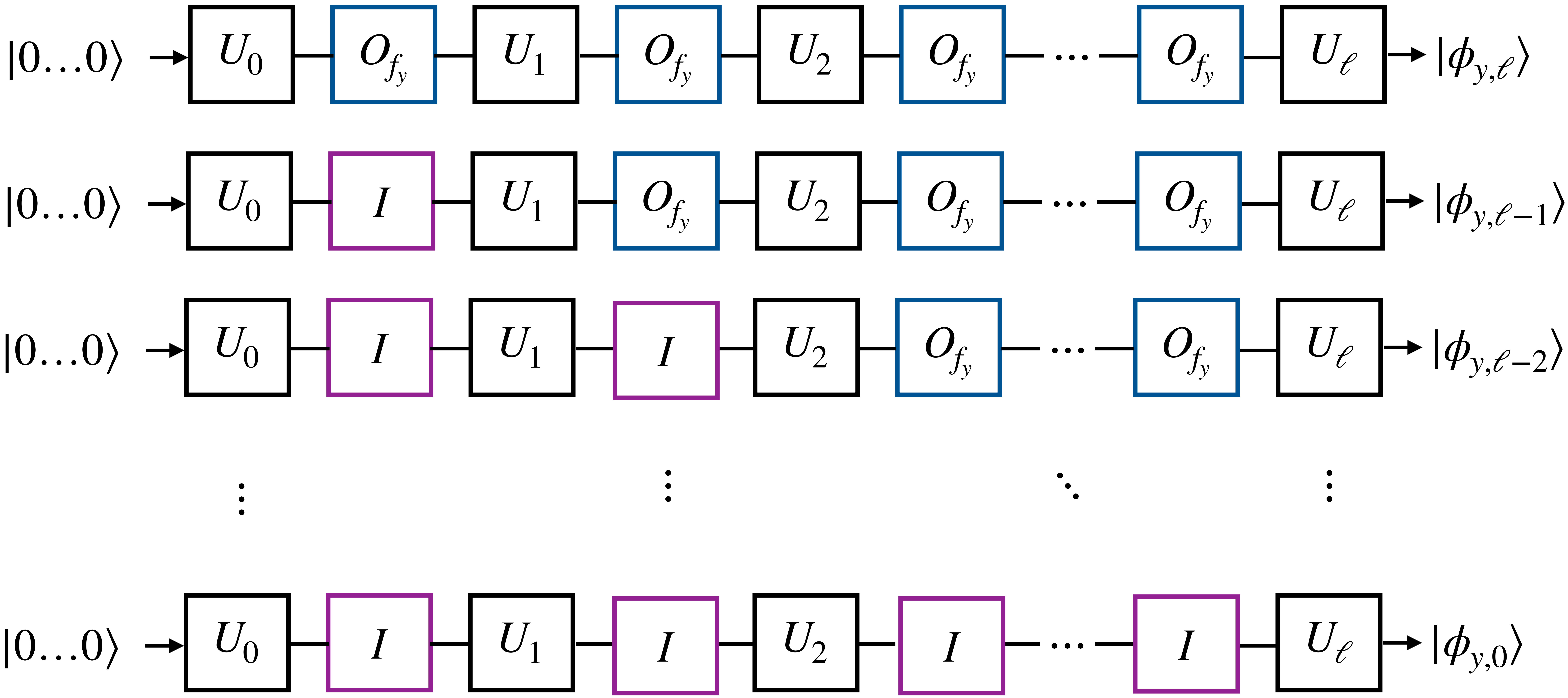

$|0\ldots0\rangle$
U_0
O_{f_y}
U_1
U_2
U_ℓ
I
$|\phi_{y,\ell}\rangle$
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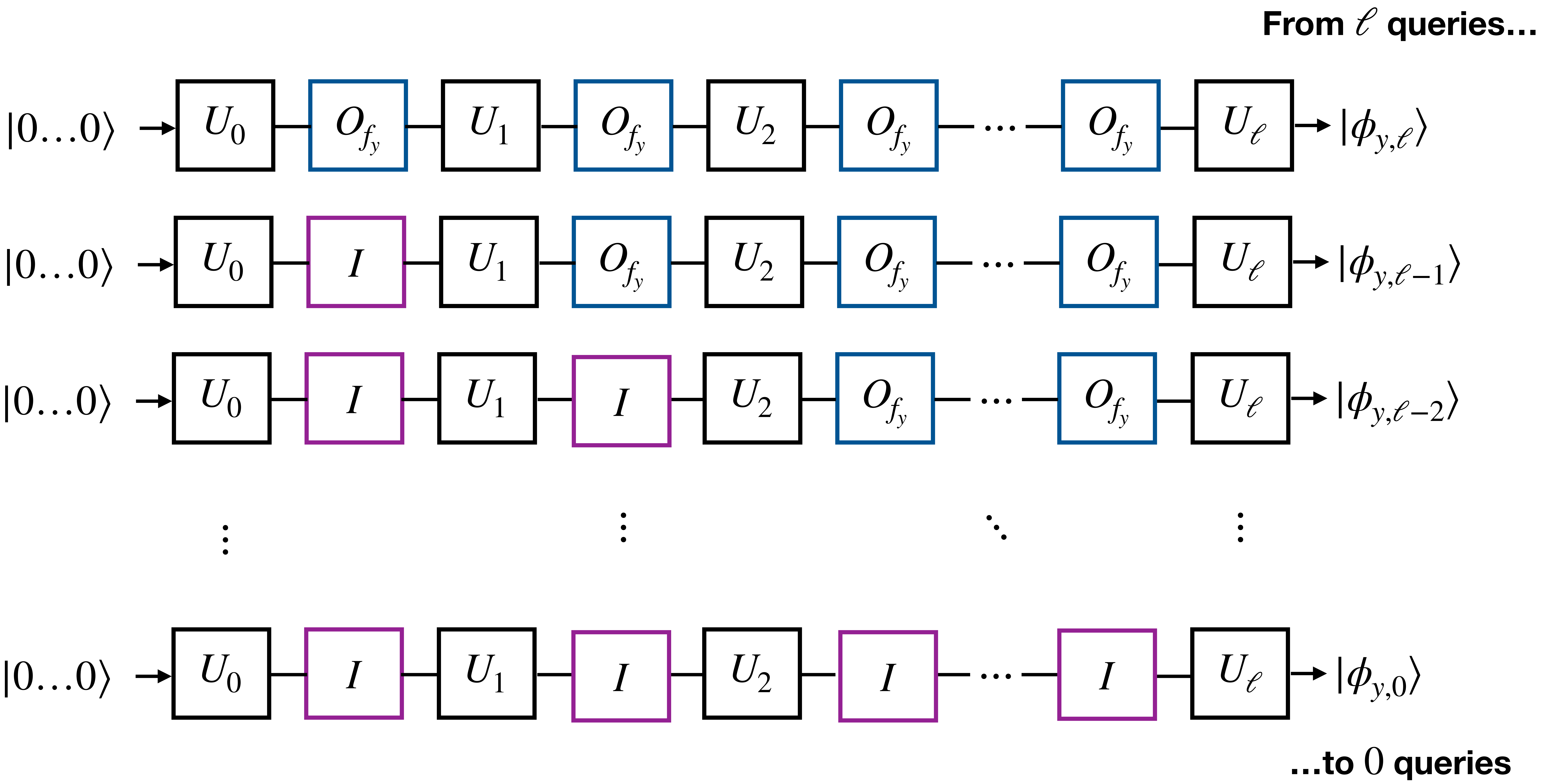
From ℓ queries...
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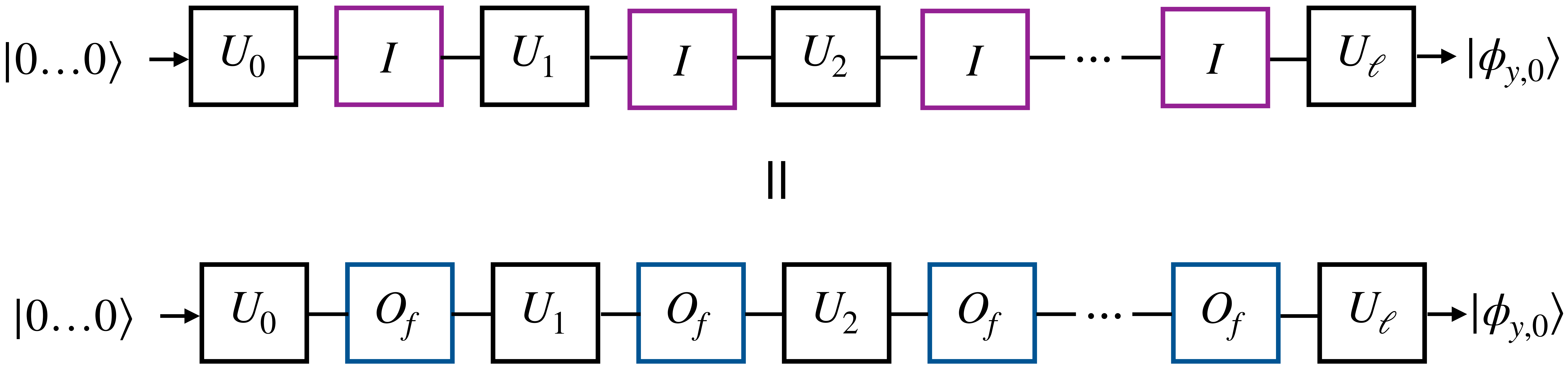
...to 0 queries

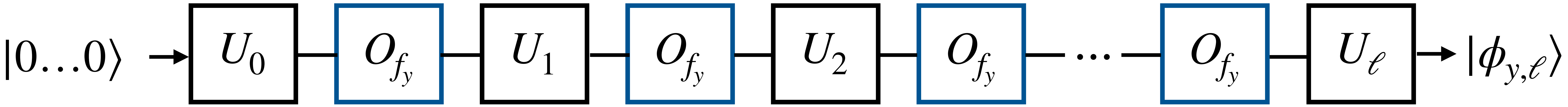
Equal Computations



where $f\colon \{0,1\}^n \to \{0,1\}$ **is the identically zero function**

Our Goal

Show that there is some choice of $y \in \{0,1\}^n$ such that the states



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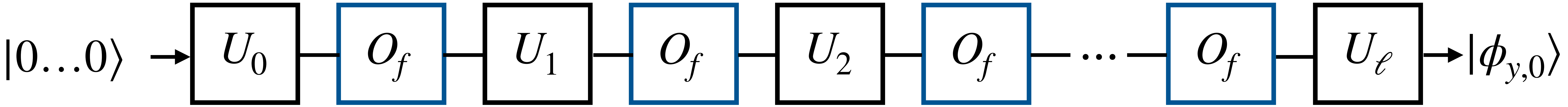

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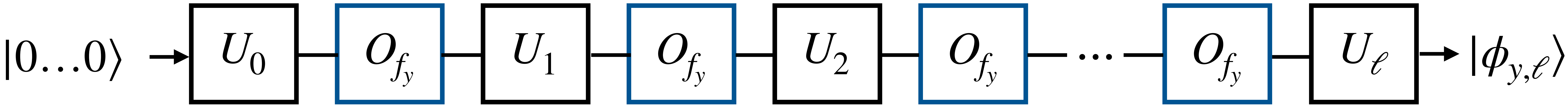

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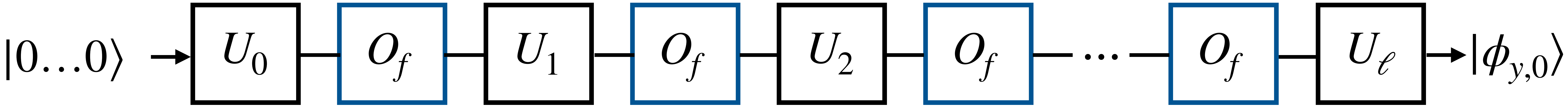

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and

$$\approx$$

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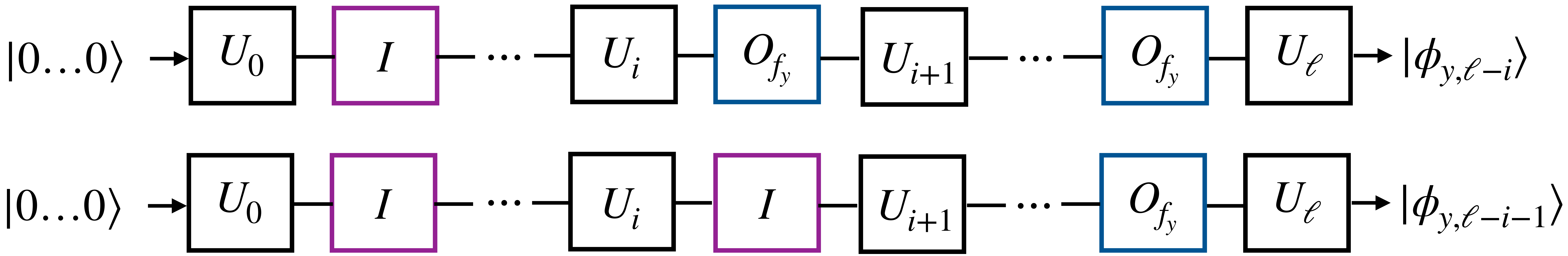
$$\Downarrow$$

The algorithm cannot distinguish well between the oracles $\boxed{O_{f_y}}$ and $\boxed{O_f}$

Consider Two Consecutive States



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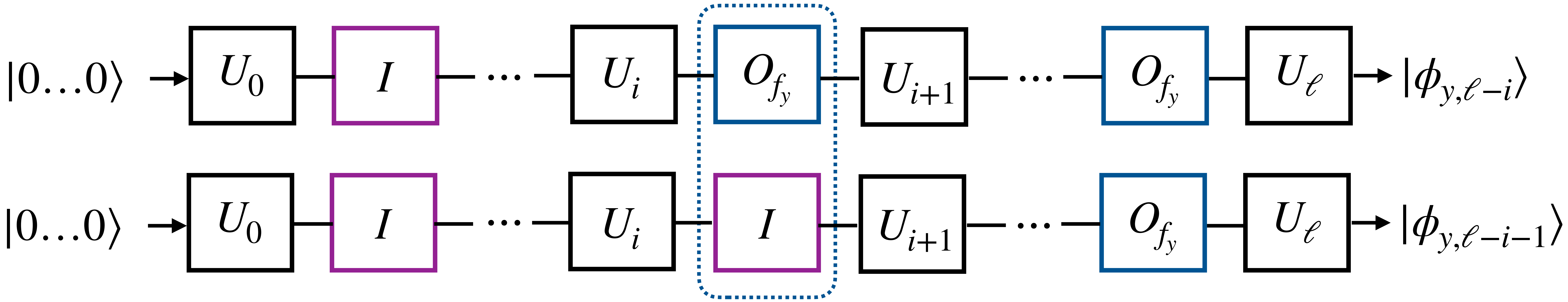

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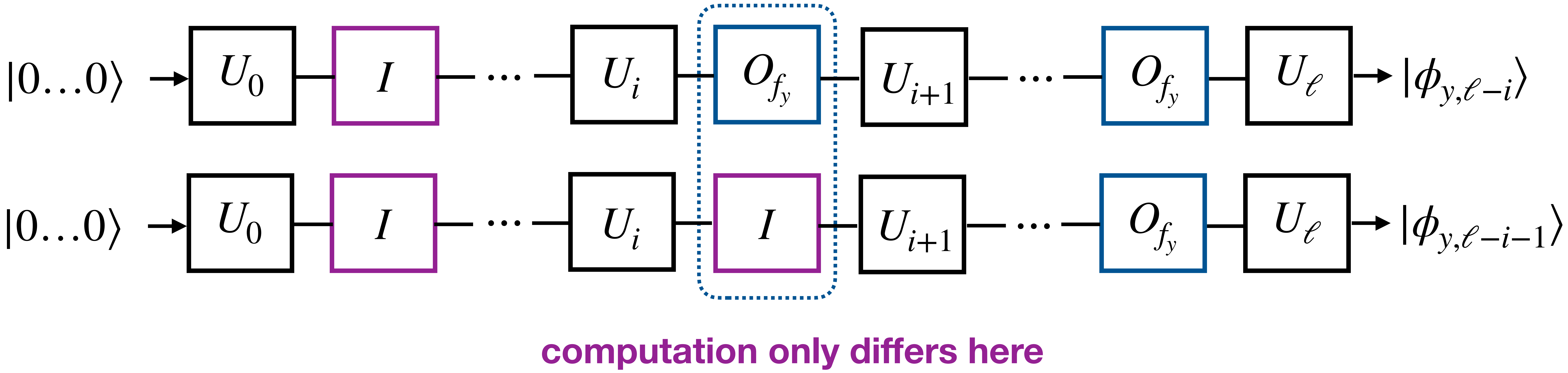
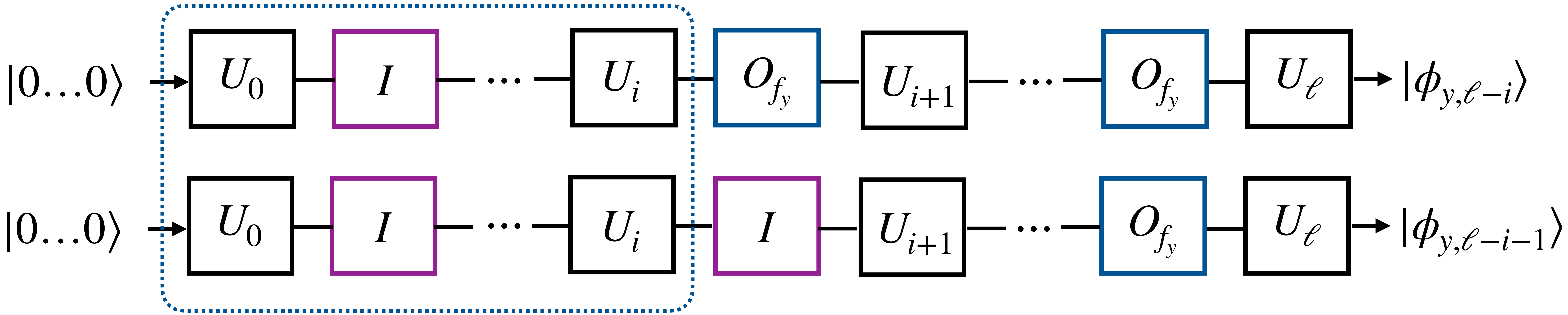


Same state up to this point



$$|\psi_i\rangle = U_i \cdots U_0 |0\ldots0\rangle = \sum_{x\in\{0,1\}^n} \alpha_{i,x} |x\rangle$$

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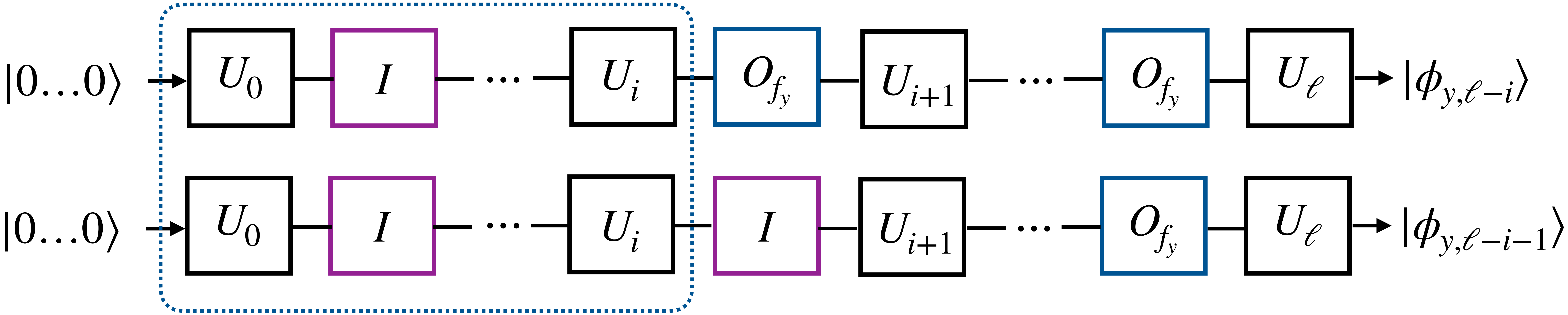

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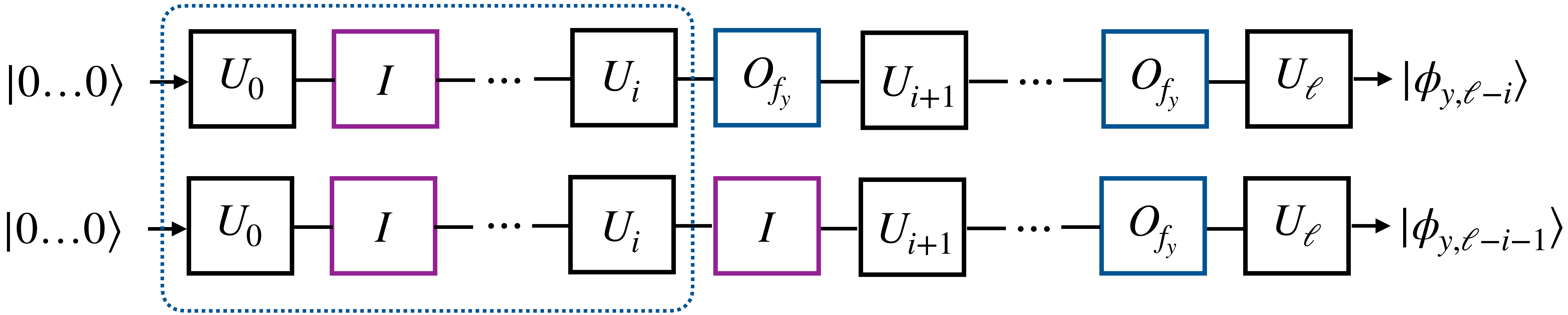

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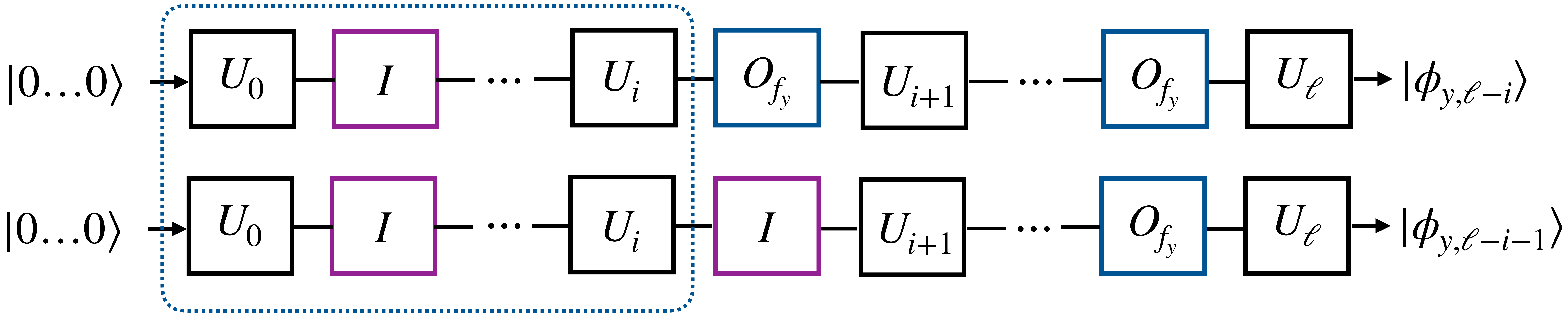

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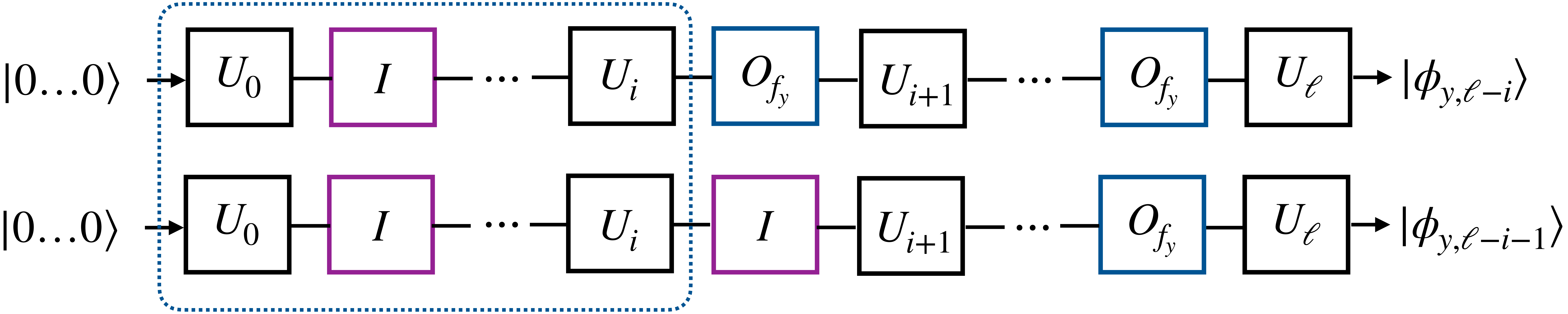

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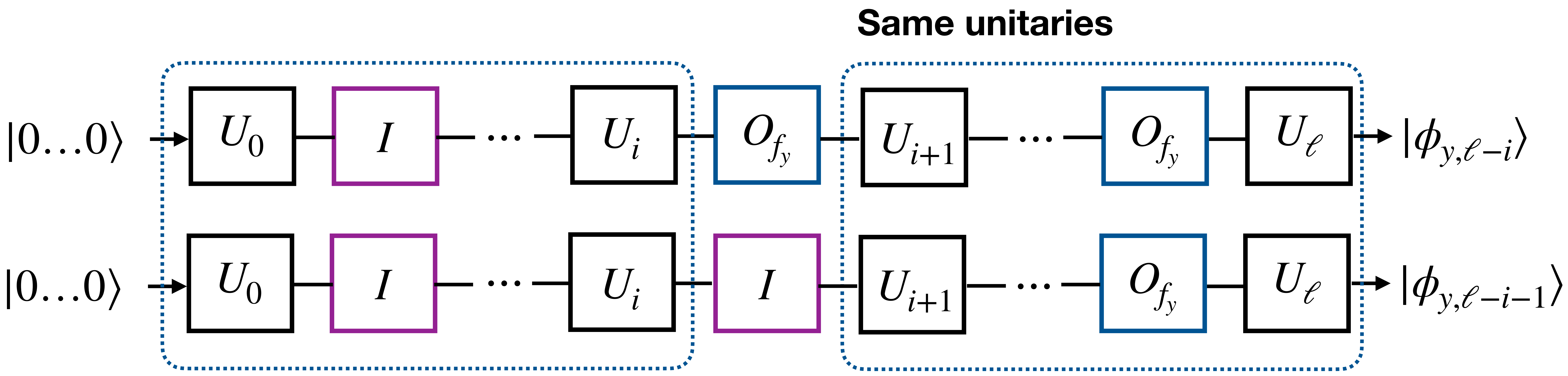

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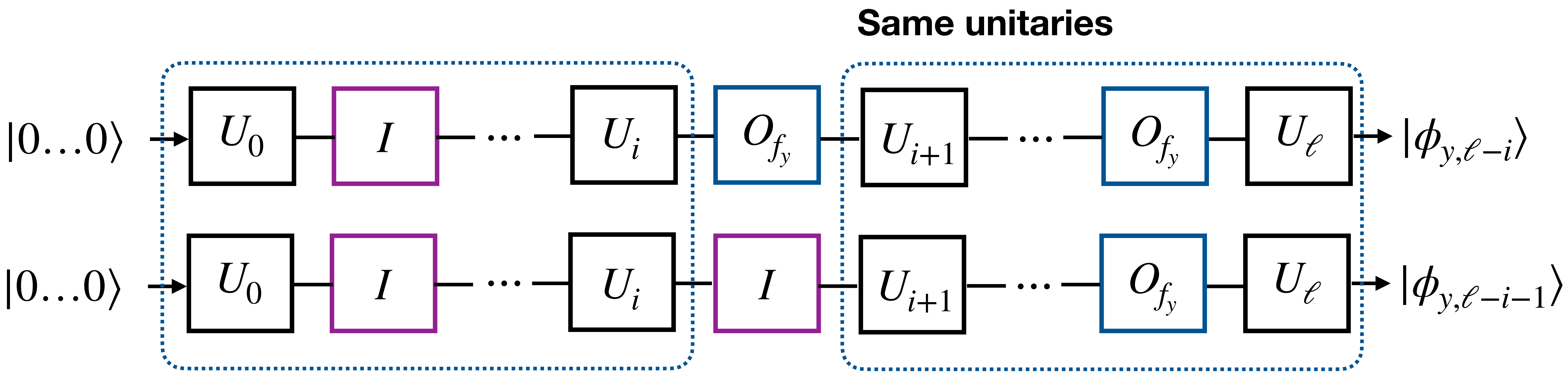

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By Triangle inequality,

$$\left\| |\phi_{y,\ell}\rangle - |\phi_{y,0}\rangle \right\|_2 \leq \left\| |\phi_{y,\ell}\rangle - |\phi_{y,\ell-1}\rangle \right\|_2 + \left\| |\phi_{y,\ell-1}\rangle - |\phi_{y,\ell-2}\rangle \right\|_2 + \cdots + \left\| |\phi_{y,1}\rangle - |\phi_{y,0}\rangle \right\|_2$$

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$\ell \geq \Omega(\sqrt{2^n})$ **needed to distinguish these states with constant probability!**

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Grover's algorithm is best possible in the number of queries!

Are there more limitations to quantum computers?

How powerful are efficient quantum computers (the class BQP)?

Complexity Classes 101

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We will first give an overview of some important complexity classes

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https://complexityzoo.net/Complexity_Zoo

(Great resource to navigate the Zoo of complexity classes)

Classical Complexity Classes 101

P

Classical Complexity Classes 101

P	**The class of decision problems solvable by a deterministic polynomial time Turing machine**

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Examples:

Classical Complexity Classes 101

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Examples: **Given a graph and two vertices, are these vertices connected?**

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DFS/BFS algorithm

Classical Complexity Classes 101

P The class of decision problems solvable by a deterministic polynomial time Turing machine

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"Efficiently solvable (classically)"

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Quantum Complexity Classes 101

BQP

Quantum Complexity Classes 101

BQP — **The class of decision problems solvable by a polynomial time quantum machine**

Quantum Complexity Classes 101

BQP

“Efficiently solvable (quantumly), the quantum version of P”

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All problems in P

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Classical computation can be made reversible with at most a polynomial overhead

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A decision version of Factoring (Shor’s algorithm)

Classical Complexity Classes 101

NP — **The class of decision problems having polynomial size proofs that can be verified by a polynomial-time deterministic Turing machine**

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We can simply ignore the proof and run the P machine

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Given the description of a boolean circuit, does it have an input that evaluates to 1?

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Circuit-SAT (well-known NP-complete problem)

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QMA

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QMA **The class of (promise) problems having polynomial-size quantum proofs that can be verified by a BQP machine**

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QMA The class of (promise) problems having polynomial-size quantum proofs that can be verified by a BQP machine

Examples:

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We can simply ignore the quantum proof and run the BQP machine

Given a local Hamiltonian and values a and b (with $b - a \geq 1/\text{poly}(n)$),
is the ground state energy below a or above b?

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(this problem requires some time to be properly introduced)

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Quantum Merlin-Arthur

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Classical Complexity Classes 101

PSPACE

Classical Complexity Classes 101

PSPACE **The class of decision problems solvable by <u>polynomial space</u> Turing machines**

Classical Complexity Classes 101

PSPACE The class of decision problems solvable by polynomial space Turing machines

EXP

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Classical Complexity Classes 101

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BPP

Classical Complexity Classes 101

PSPACE The class of decision problems solvable by polynomial space Turing machines

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BPP The class of decision problems solvable by probabilistic polynomial time Turing machines

Classical Complexity Classes 101

PSPACE	**The class of decision problems solvable by <u>polynomial space</u> Turing machines**
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<u>M</u>erlin-<u>A</u>rthur

Complexity Classes 101

Major open problem in Computer Science and Mathematics

P *vs* **NP**

Complexity Classes 101

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"Efficiently solvable (classically)"

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Complexity Classes 101

Major open problem in Computer Science and Mathematics

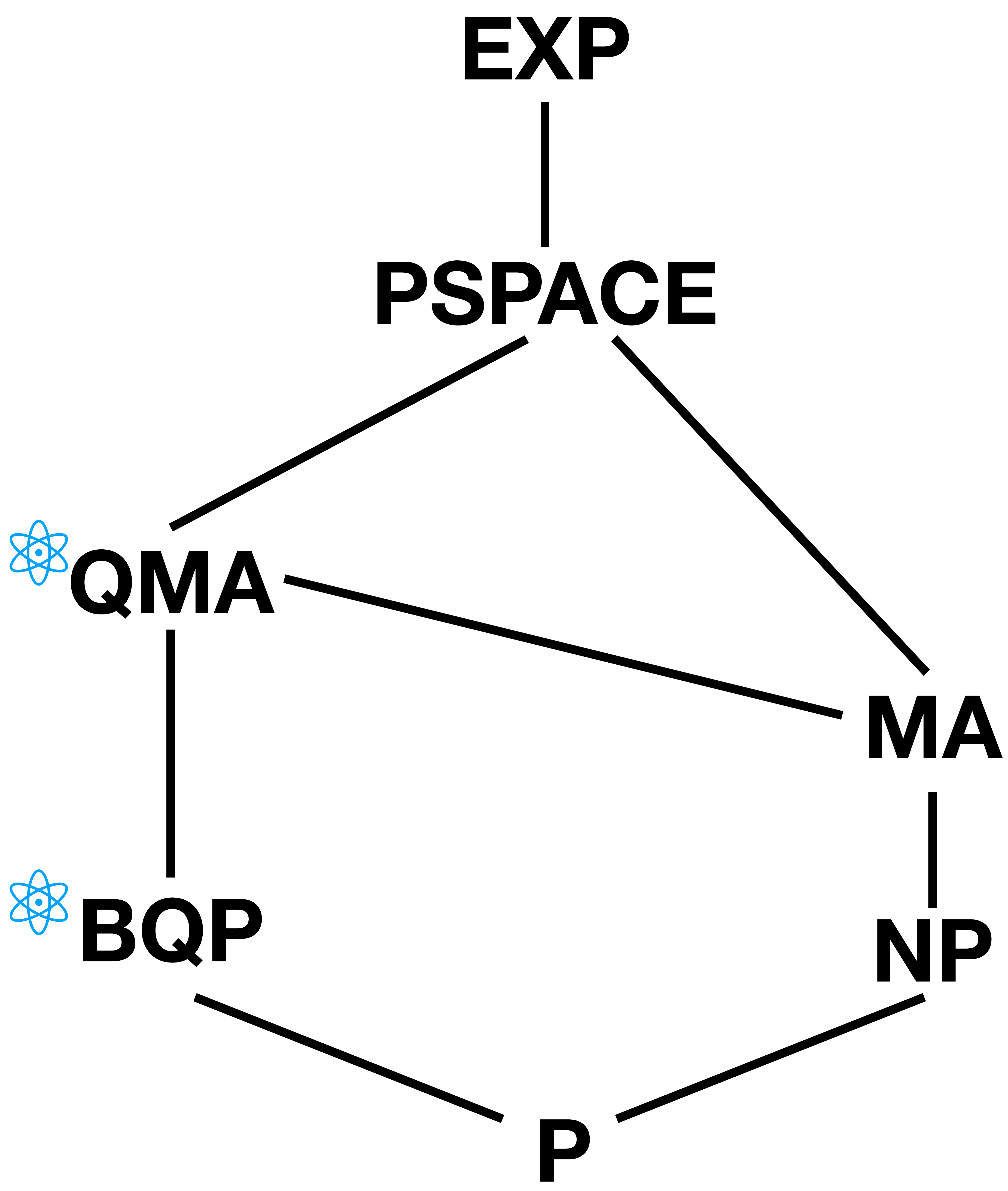
"Efficiently solvable (classically)"

P *vs* **NP**

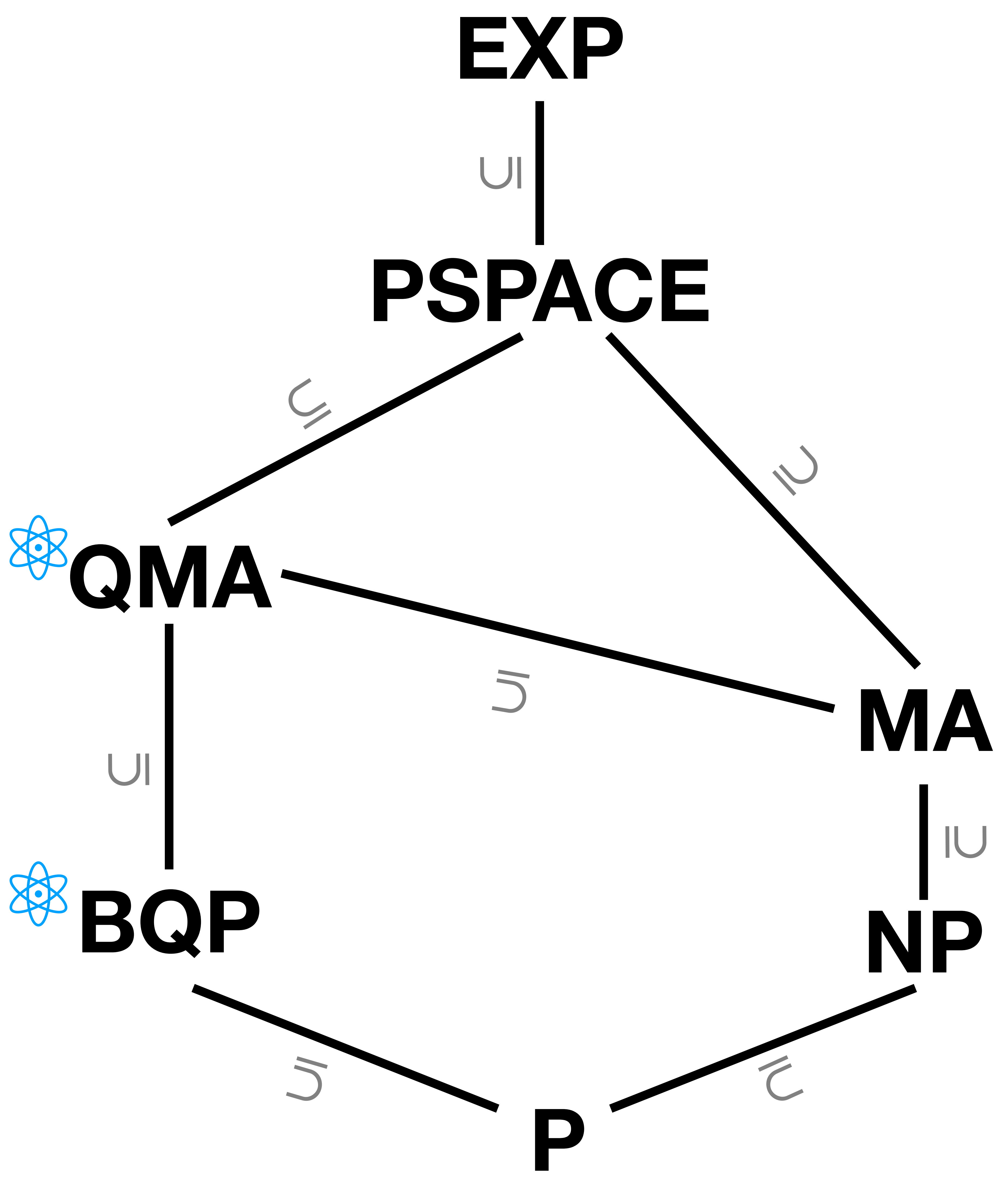
"Solution can be efficiently verified (classically)"

"Is proving a theorem harder than verifying it?"

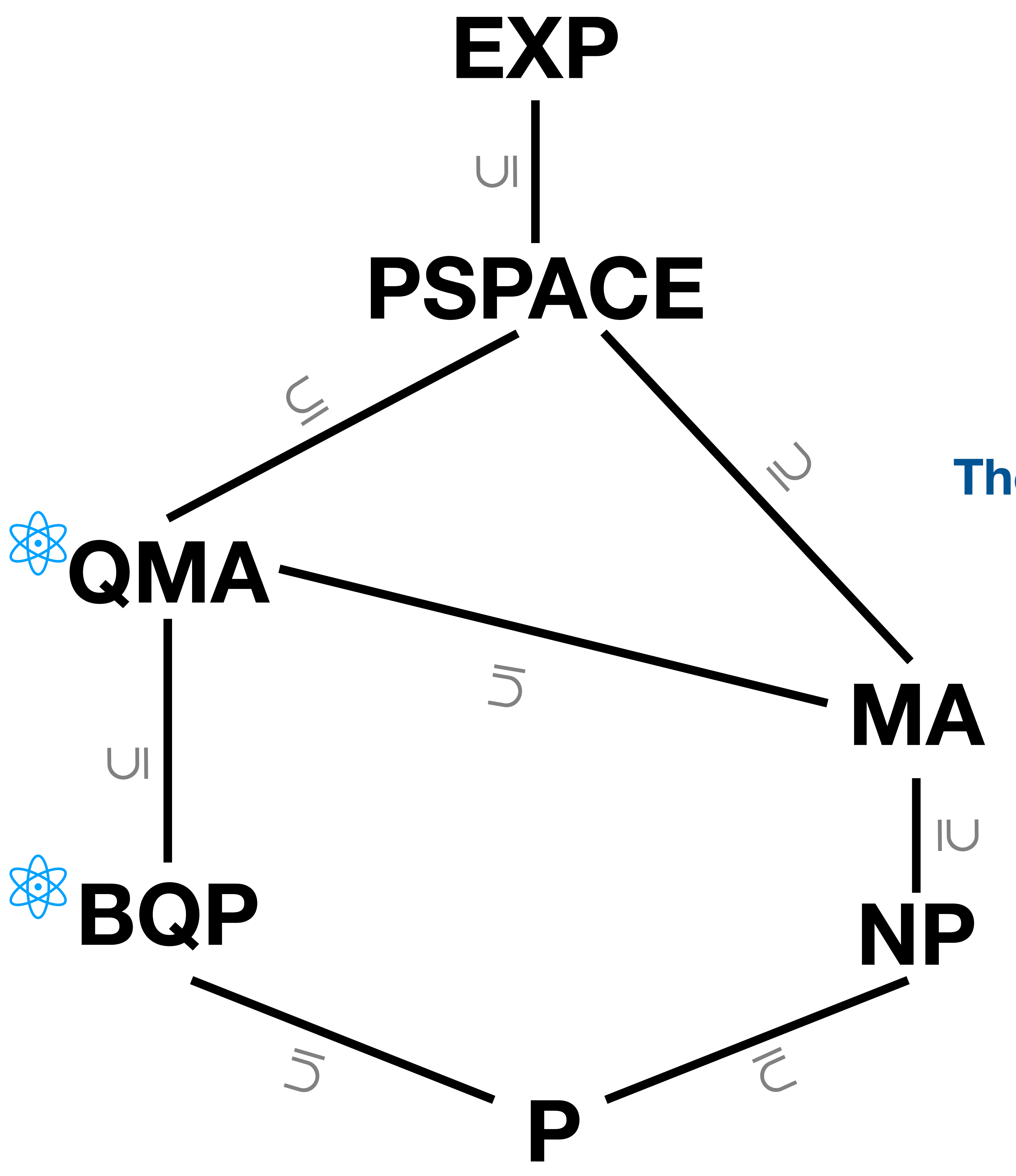
Classical and Quantum Complexity Classes



Classical and Quantum Complexity Classes

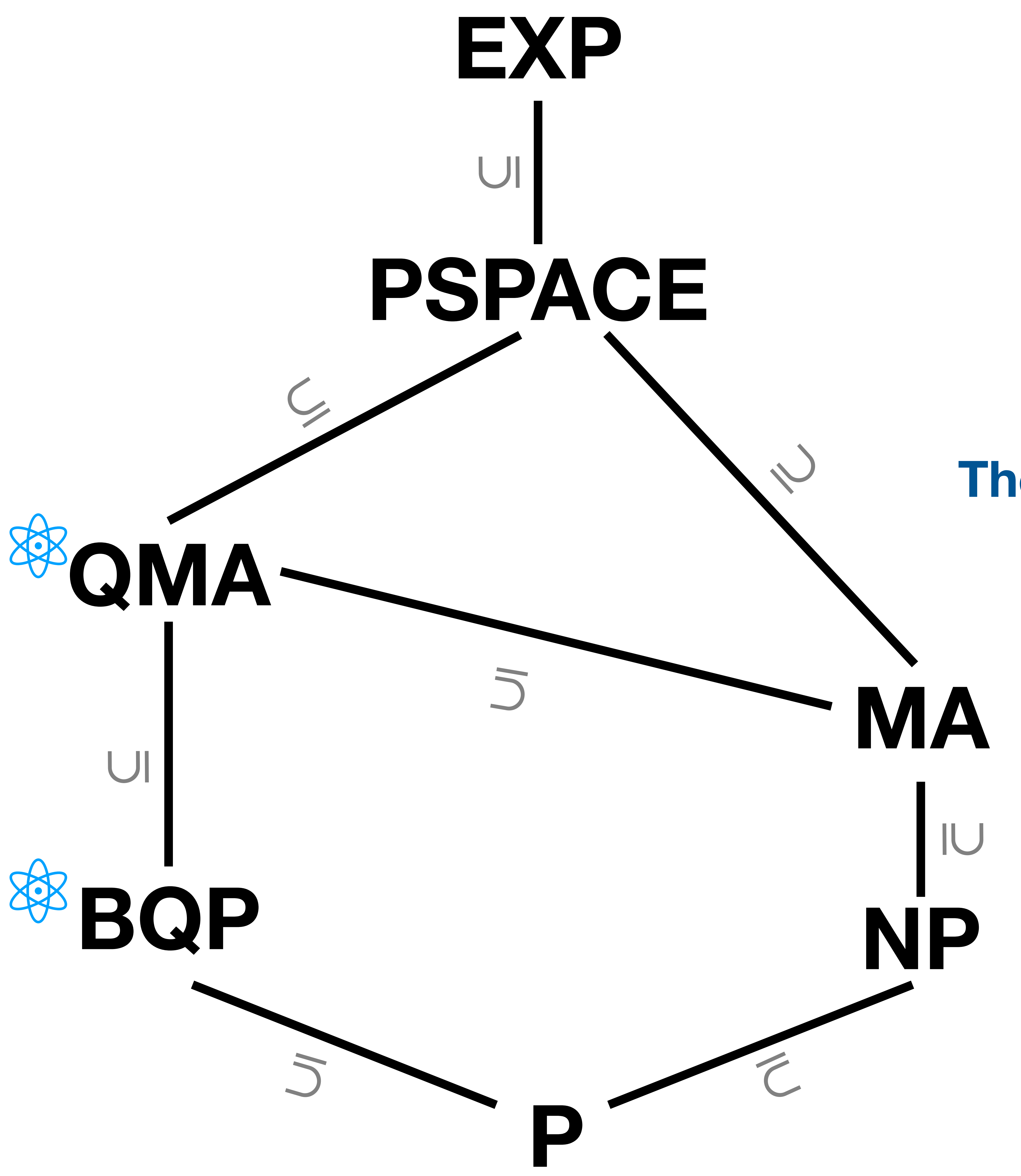


Classical and Quantum Complexity Classes



These inclusions were rigorously proved!

Classical and Quantum Complexity Classes



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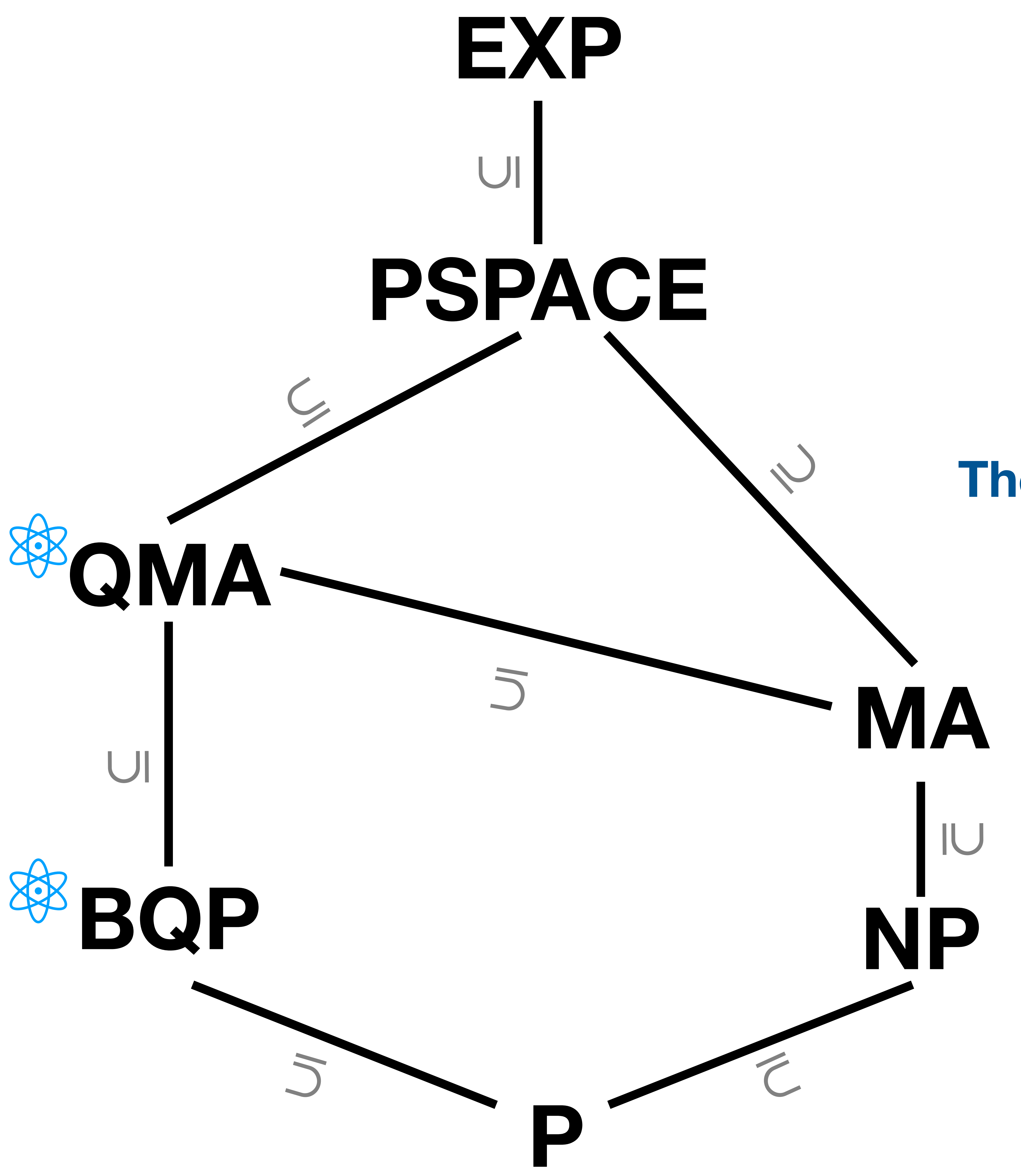
BQP and NP are conjectured to be incomparable!

Classical and Quantum Complexity Classes



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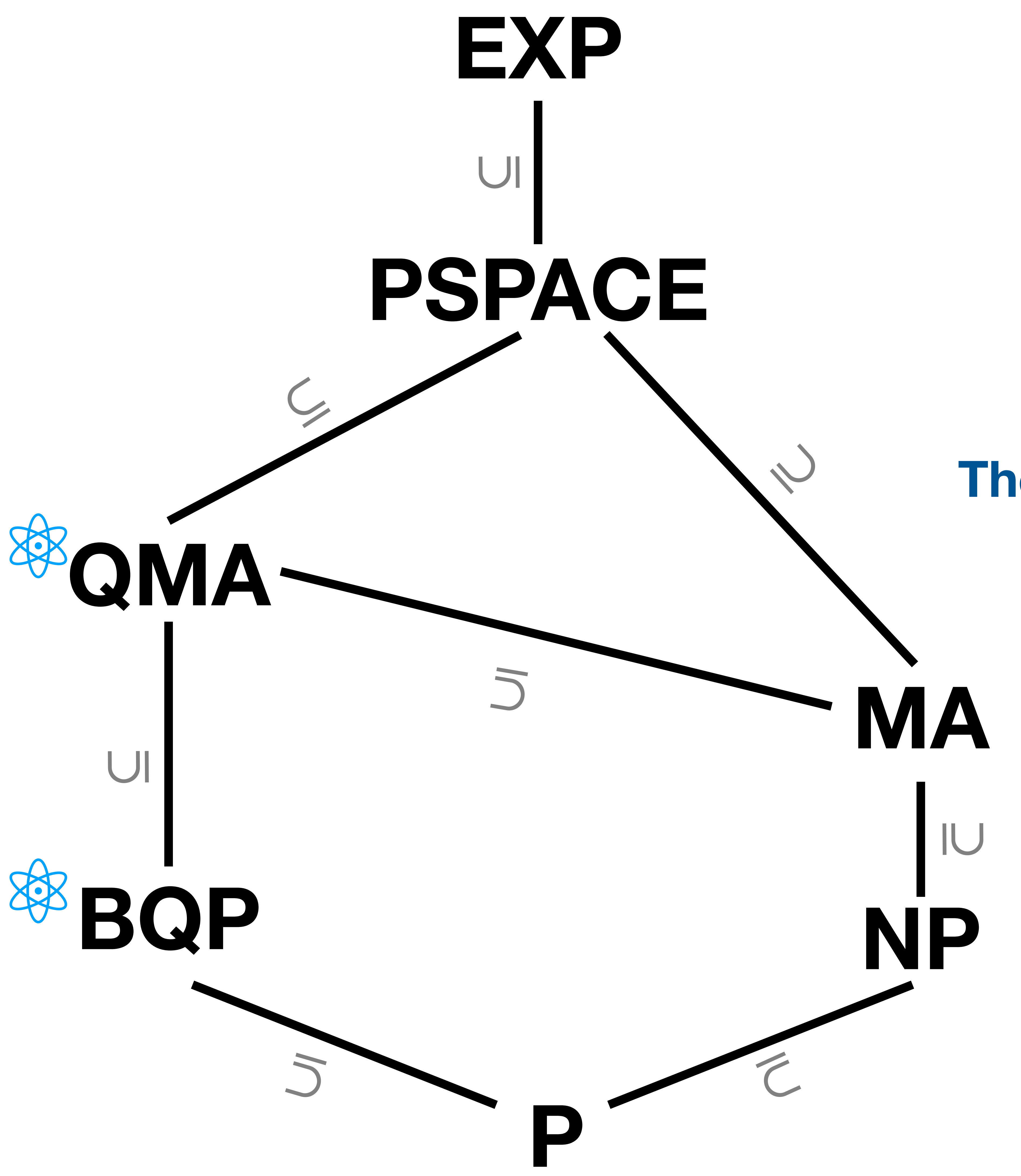
BQP and NP are conjectured to be incomparable!

NP $\not\subseteq$ BQP

Classical and Quantum Complexity Classes



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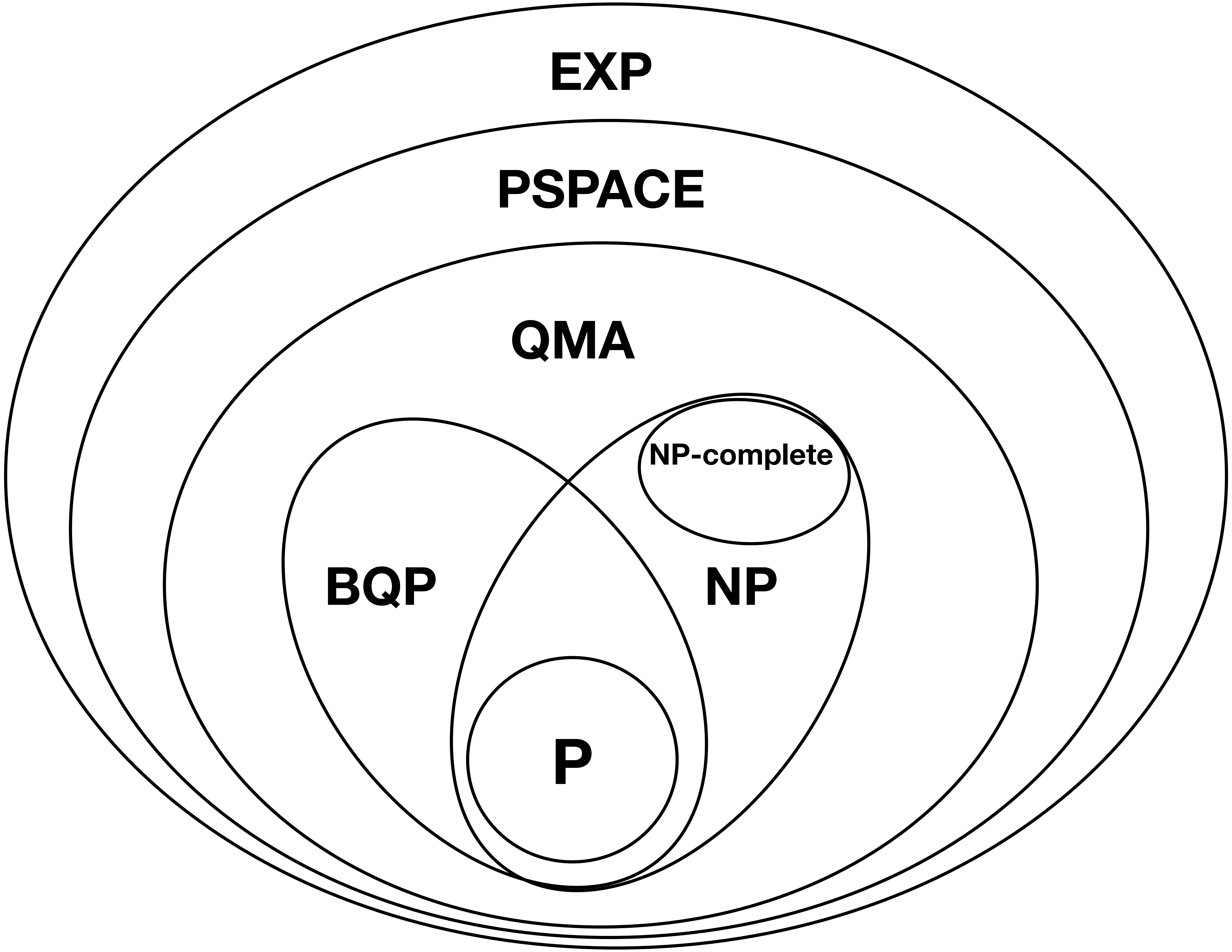
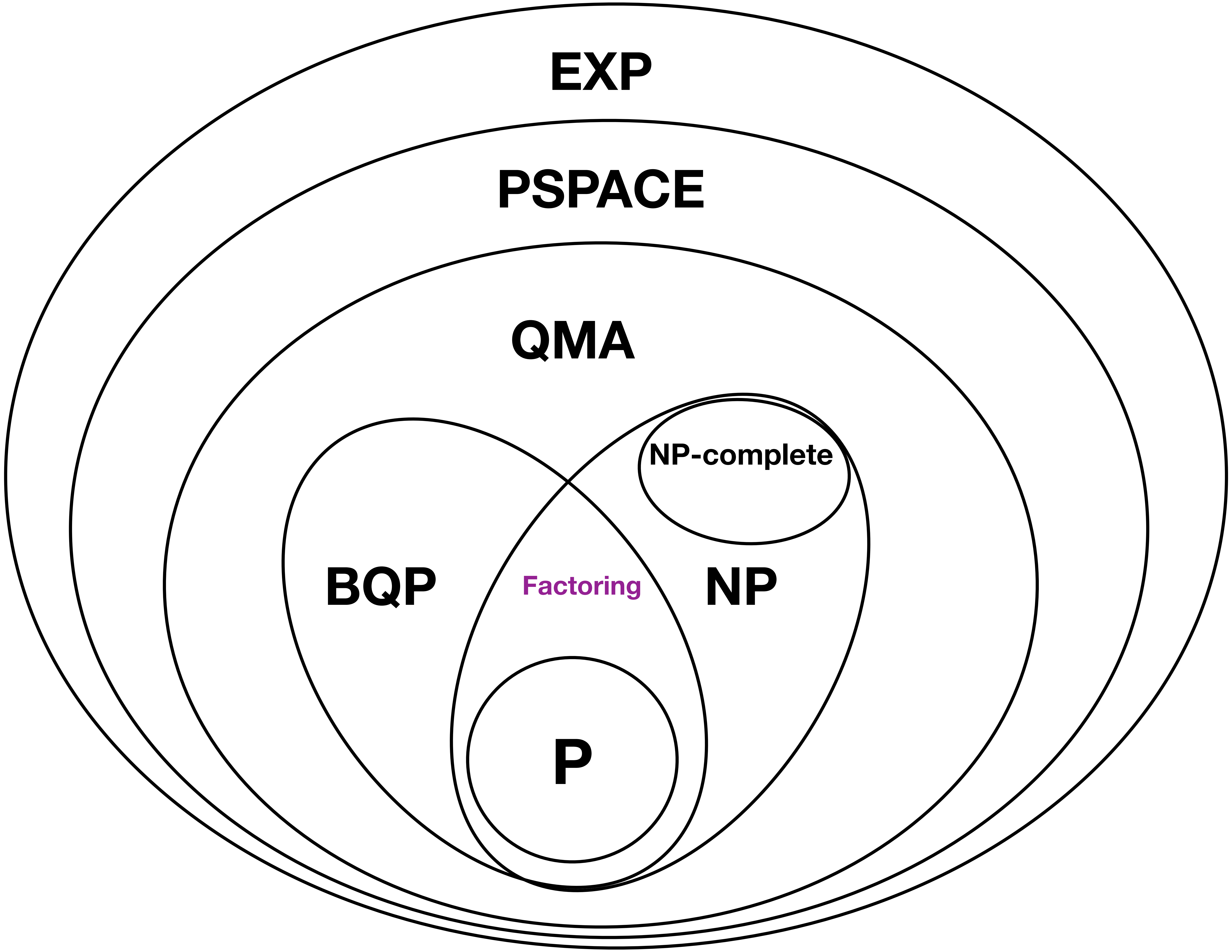
Conjectured Complexity Landscape



<u>Conjectured</u> Complexity Landscape



Can we give a complexity upper bound on BQP?

Warmup

EXP

$\subseteq$

BQP

Warmup

EXP

BQP

<u>Implication</u>: efficient quantum computers will never provide more than some exponential speed-ups (if any) over classical ones

A Better Upper Bound on BQP

PSPACE

$\rotatebox{90}{$\subseteq$}$

BQP

Proving the Upper Bound

Given $x \in \{0,1\}^n, \quad x \overset{?}{\in} L$ **(where L is language in BQP)**

Proving the Upper Bound

Given $x \in \{0,1\}^n, \quad x \overset{?}{\in} L$ (where L is language in BQP)

$$U_m U_{m-1} \cdots U_2 U_1 |\underbrace{0\ldots0}_{\ell}\rangle$$

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BQP verifier

$m = \textbf{poly}(n), \ell = \textbf{poly}(n)$

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Given $x \in \{0,1\}^n, \quad x \stackrel{?}{\in} L$ (where L is language in BQP)

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BQP verifier measures the first qubit and accept iff outcome is $|1\rangle$

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Can we compute a single amplitude $\alpha_{y'}$ in PSPACE?

Proving the Upper Bound

Given $x \in \{0,1\}^n, \quad x \overset{?}{\in} L$

$$\alpha_{y'} = \langle y' | U_m U_{m-1} \cdots U_2 U_1 | 0 \ldots 0 \rangle$$

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Fact: $I = \sum_{y \in \{0,1\}^\ell} |y\rangle\langle y|$

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Each factor is efficient to compute

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Each factor is efficient to compute, and there are only $m =$ poly(n)

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...

Each factor is efficient to compute, and there are only $m = \textbf{poly}(n)$

$\Longrightarrow$ summation can be computed in PSPACE

BQP verifier

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Given $x \in \{0,1\}^n, \quad x \overset{?}{\in} L$

$$\Pr[\textbf{BQP verifier accepts } x] = \sum_{y \in \{0,1\}^\ell :\ y_1 = 1} |\alpha_y|^2$$

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$\Longrightarrow$ **this probability can be computed in PSPACE**

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$\Longrightarrow$ **BQP $\subseteq$ PSPACE**

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$\implies$ **this probability can be computed in PSPACE**

$\implies$ **BQP $\subseteq$ PSPACE**

$\blacksquare$

Thank you!

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More Questions?