Unstructured Search

via Grover’s Algorithm

Fernando Granha Jeronimo

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Any classical computer requires $\Omega(2^n)$ queries!

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Can a quantum computer beat brute-force search?

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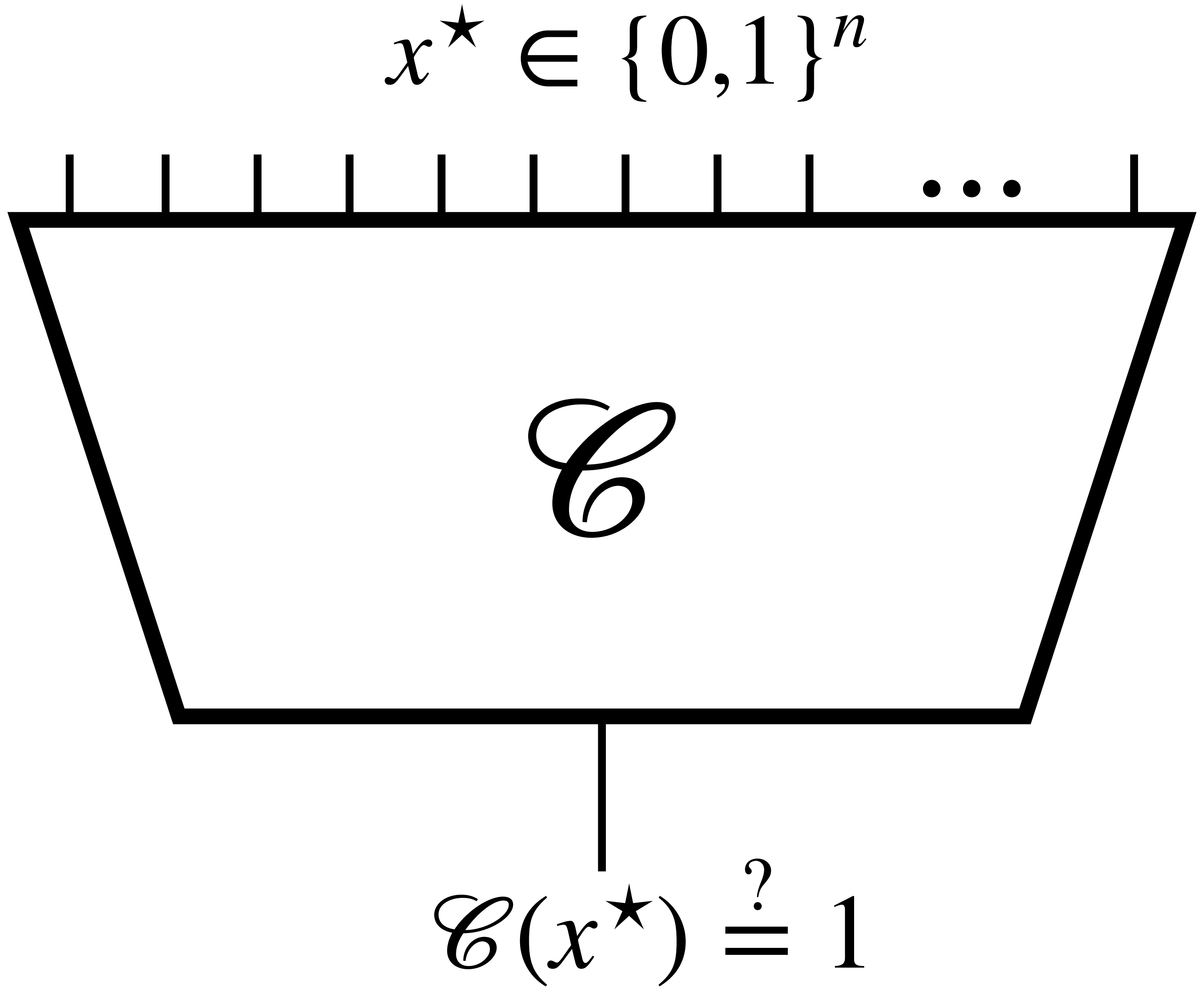

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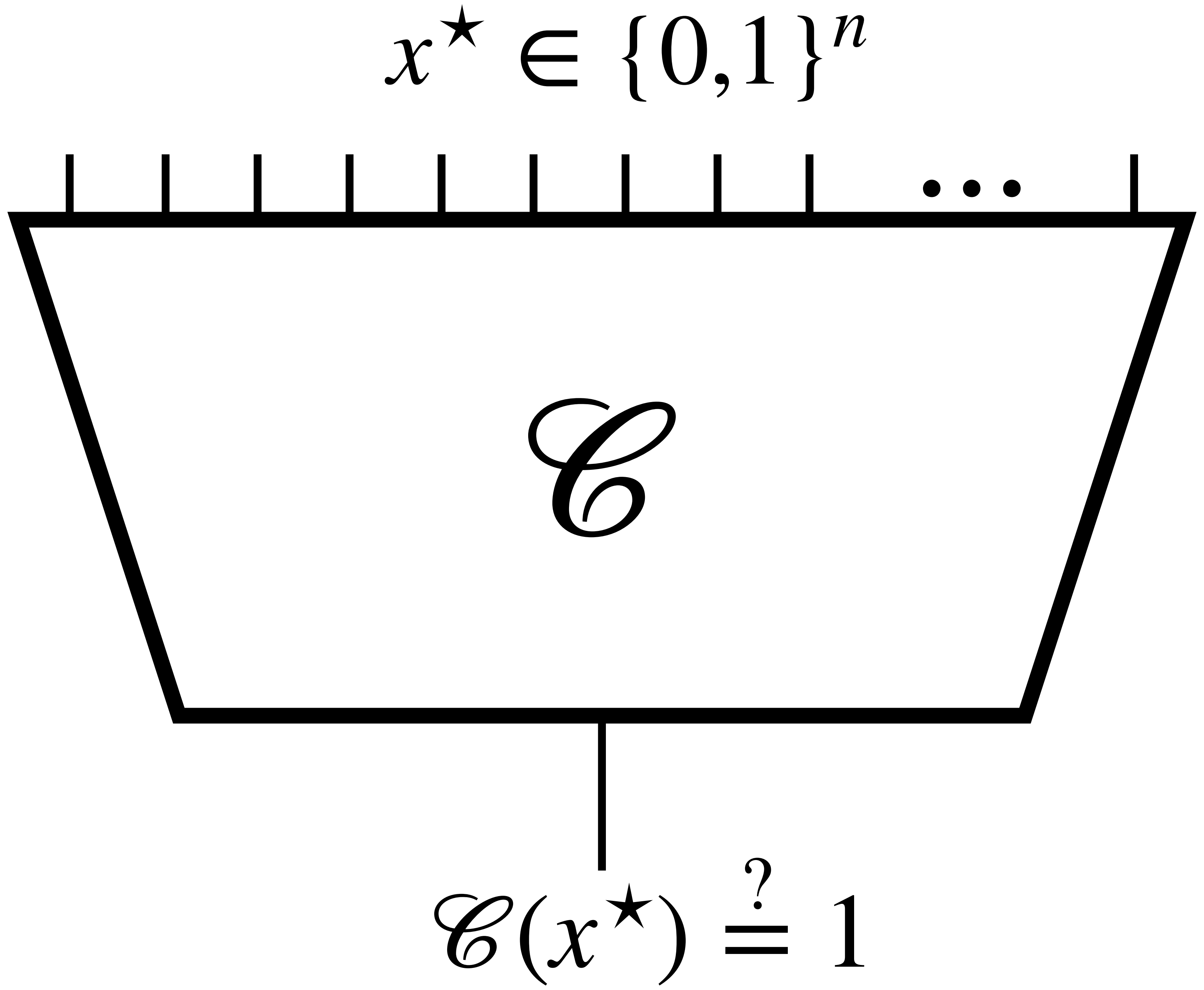

Determining if $\mathscr{C}$ has a satisfying input is NP-complete

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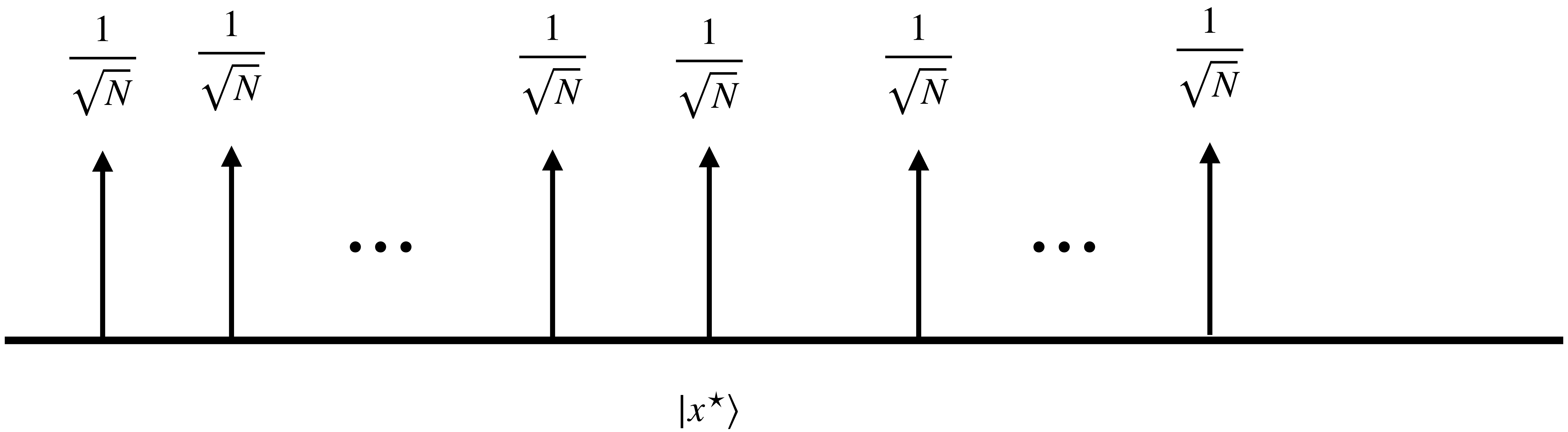
(Phase Oracle)

$$\Uparrow$$

$$O_f|x\rangle|0\rangle = |x\rangle|f(x)\rangle$$

Some Intuition

$N = 2^n$

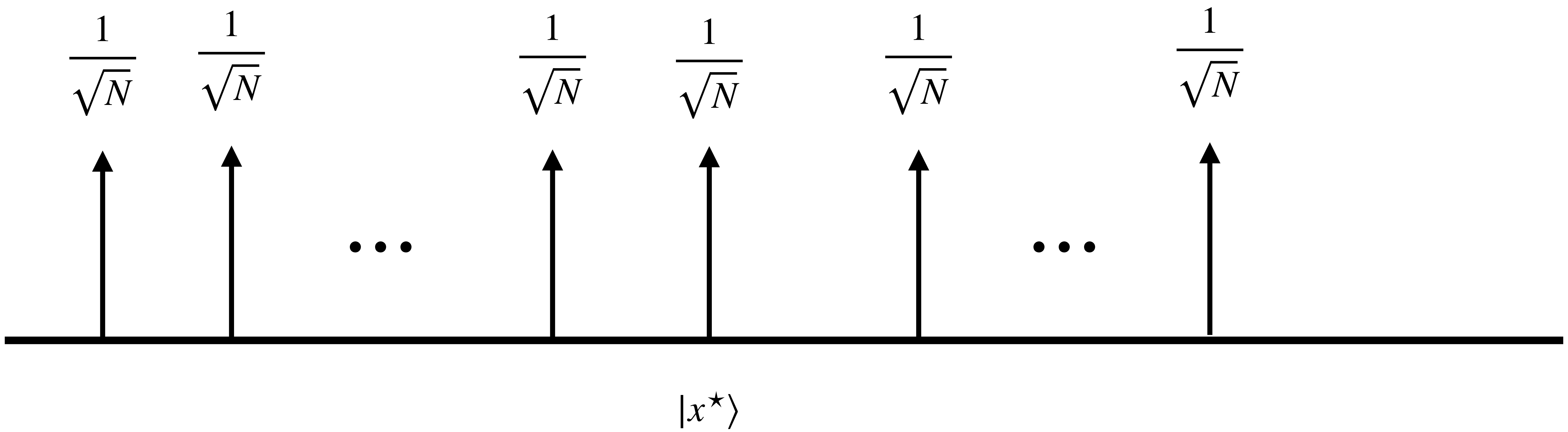


$$|u\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle$$

Some Intuition

$N = 2^n$

For now, assume there is exactly one $x^\star$ with $f(x^\star) = 1$



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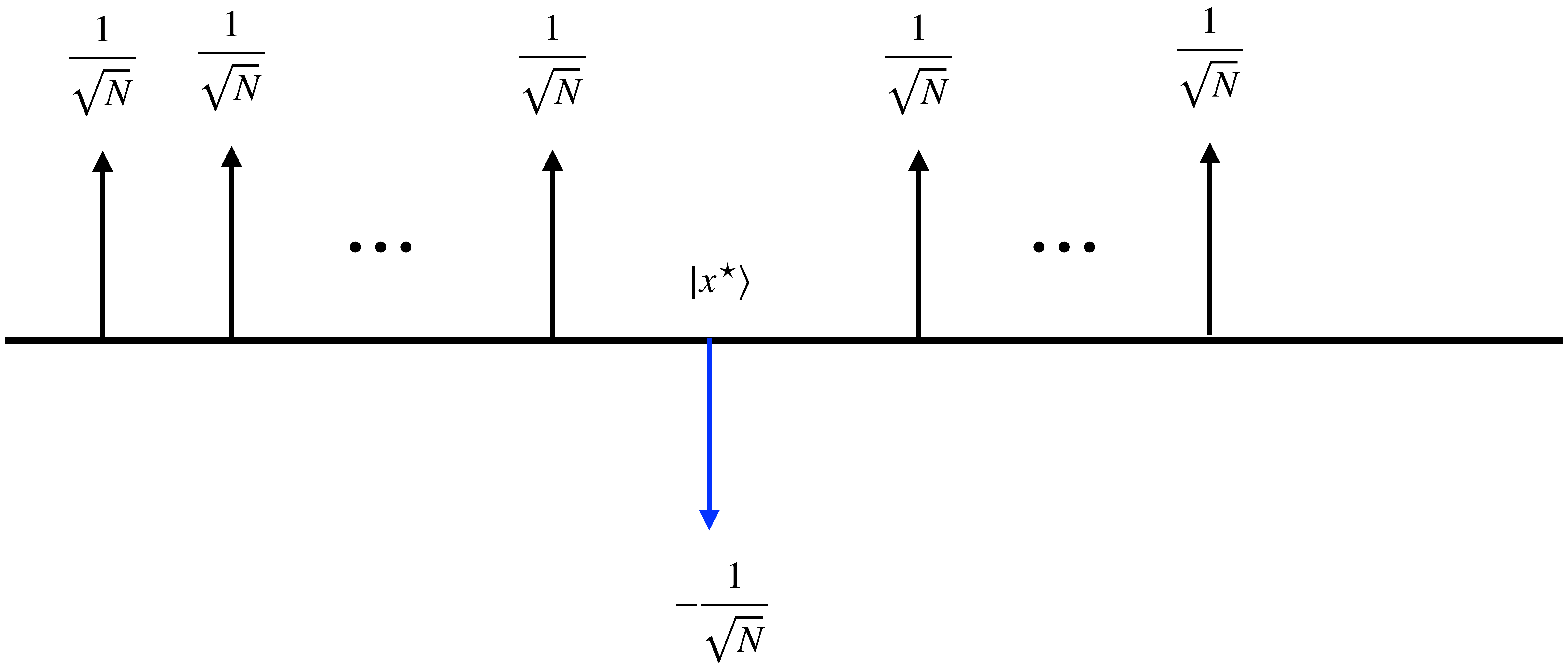

$O_f|u\rangle$

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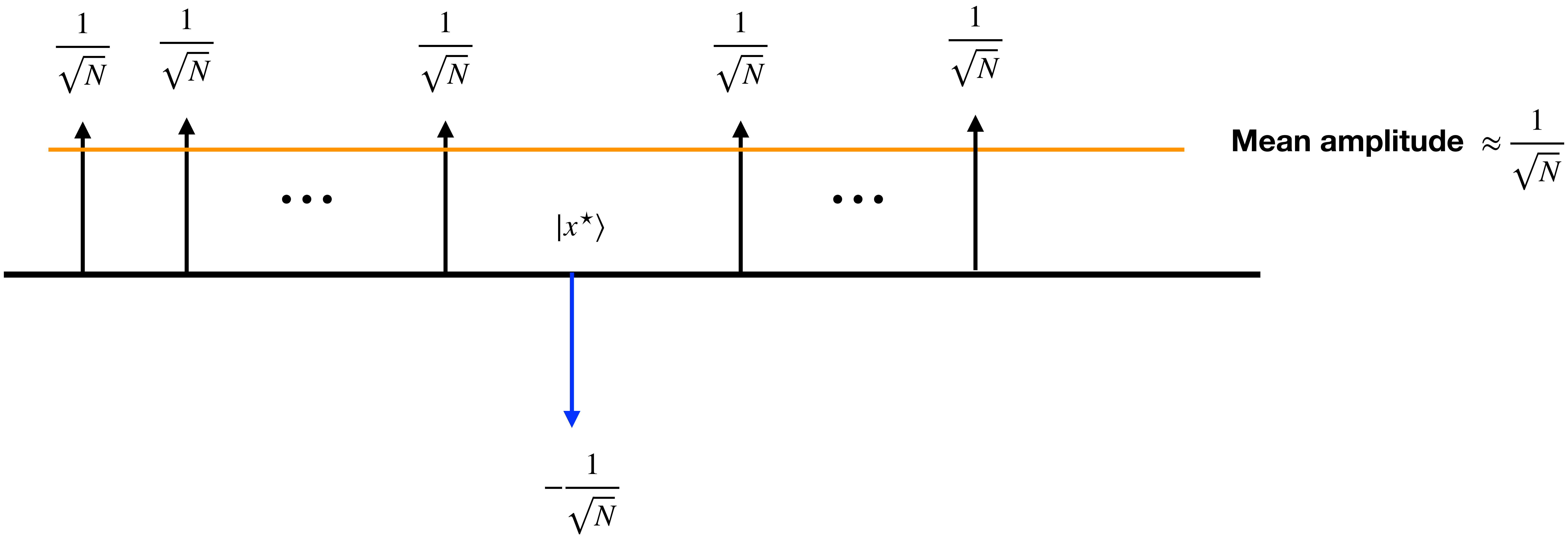

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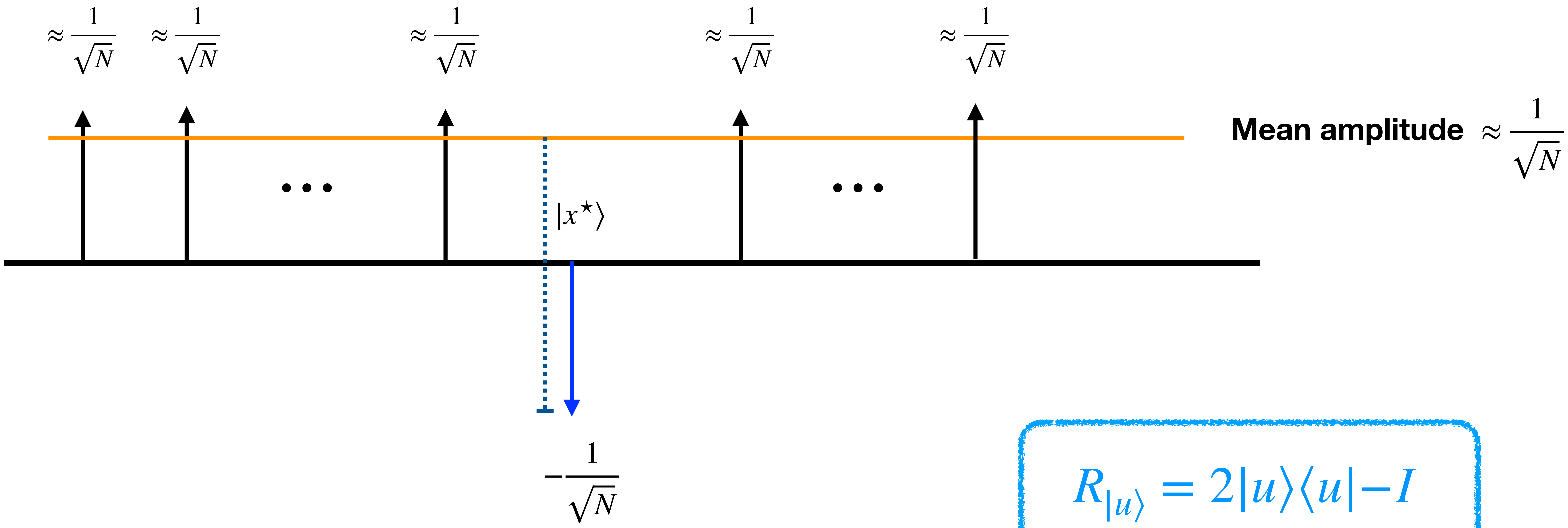

$$\boxed{R_{|u\rangle} = 2|u\rangle\langle u| - I}$$

Reflect around the "mean"

$$R_{|u\rangle} O_f |u\rangle$$

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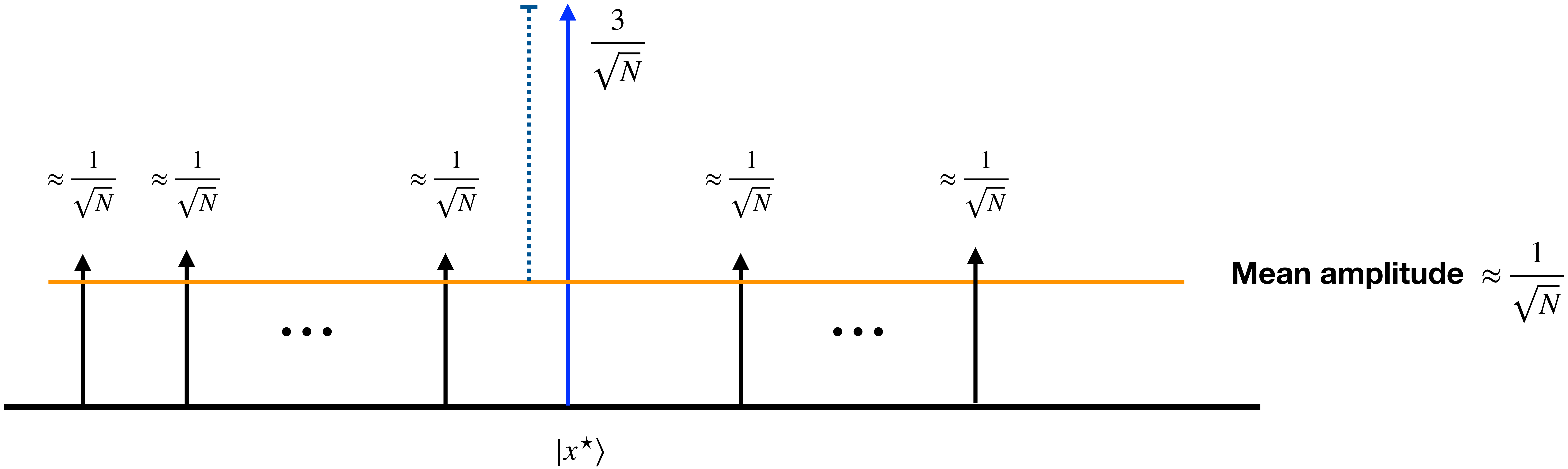

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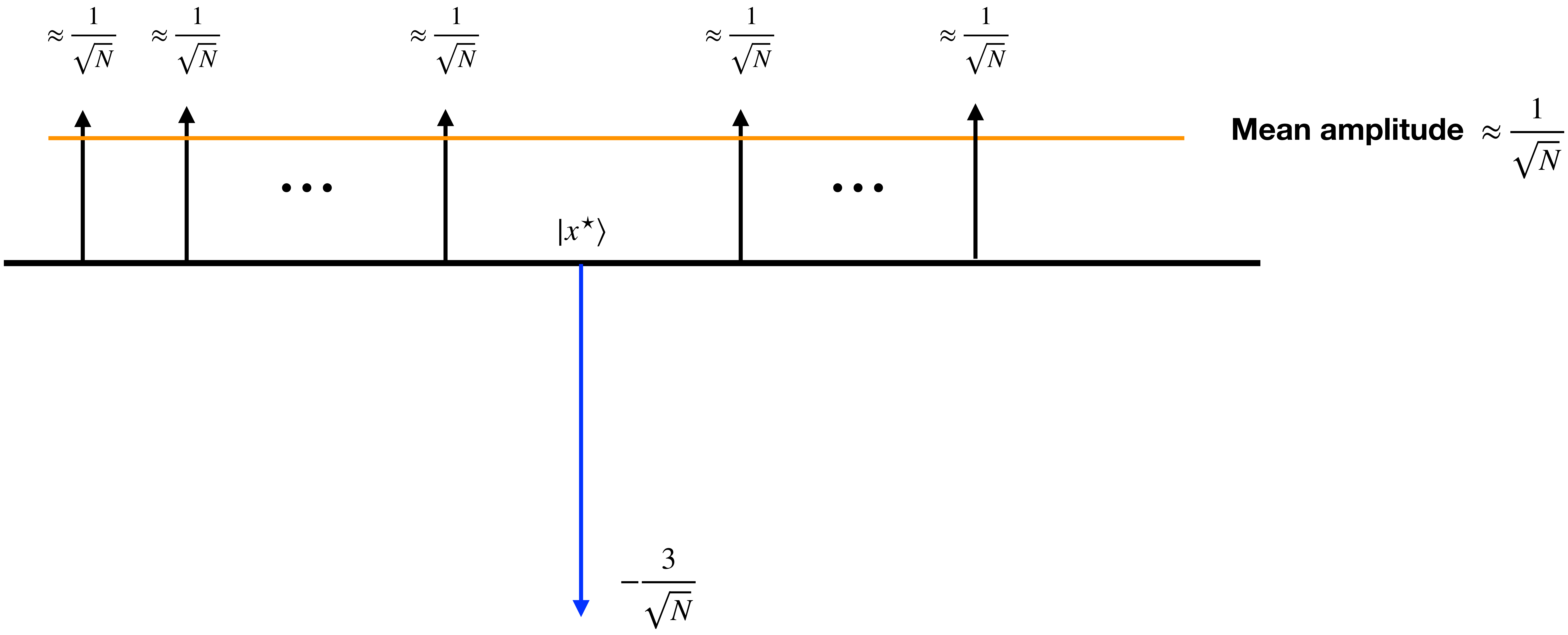

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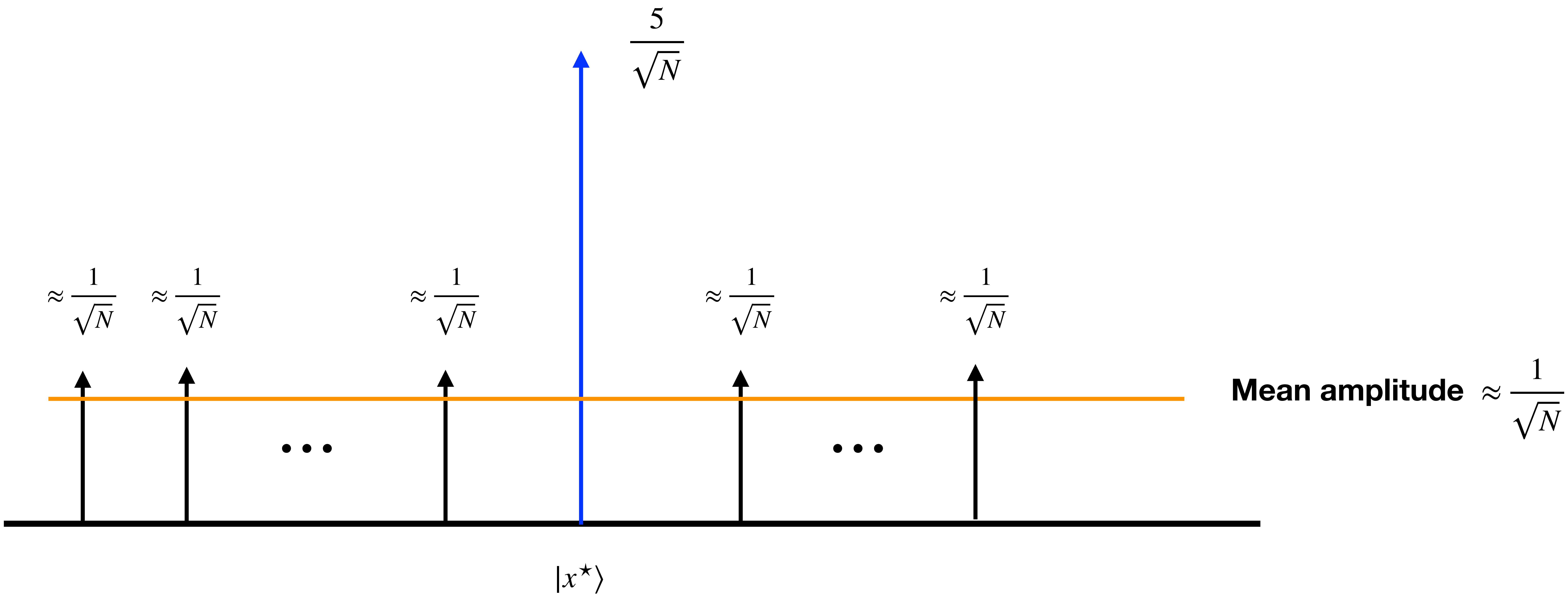

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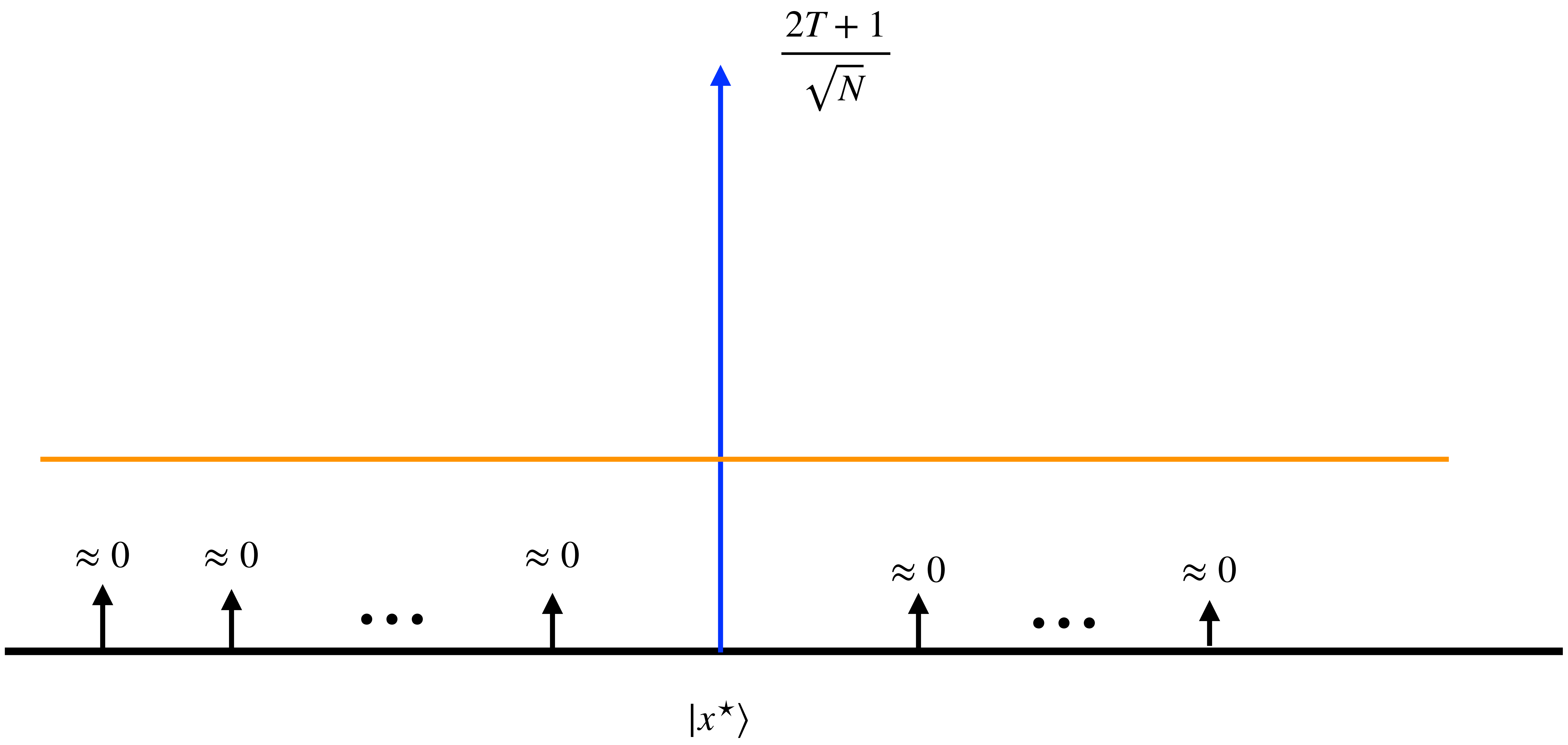
Extrapolating, we would guess…



After T queries $(R_{|u\rangle} O_f)^T |u\rangle$

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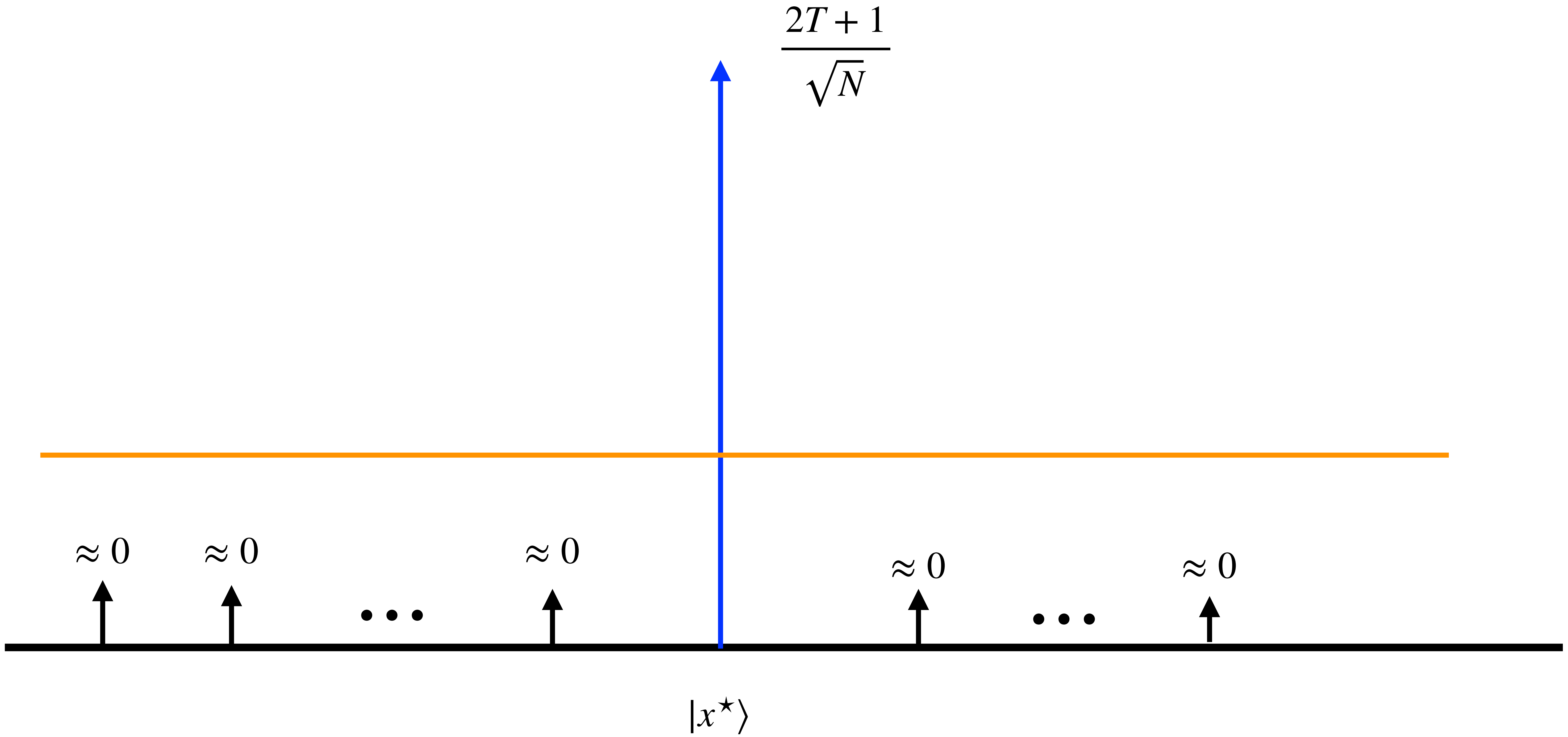
Suggests $T = O(\sqrt{N})$ suffice, beating brute-force search!



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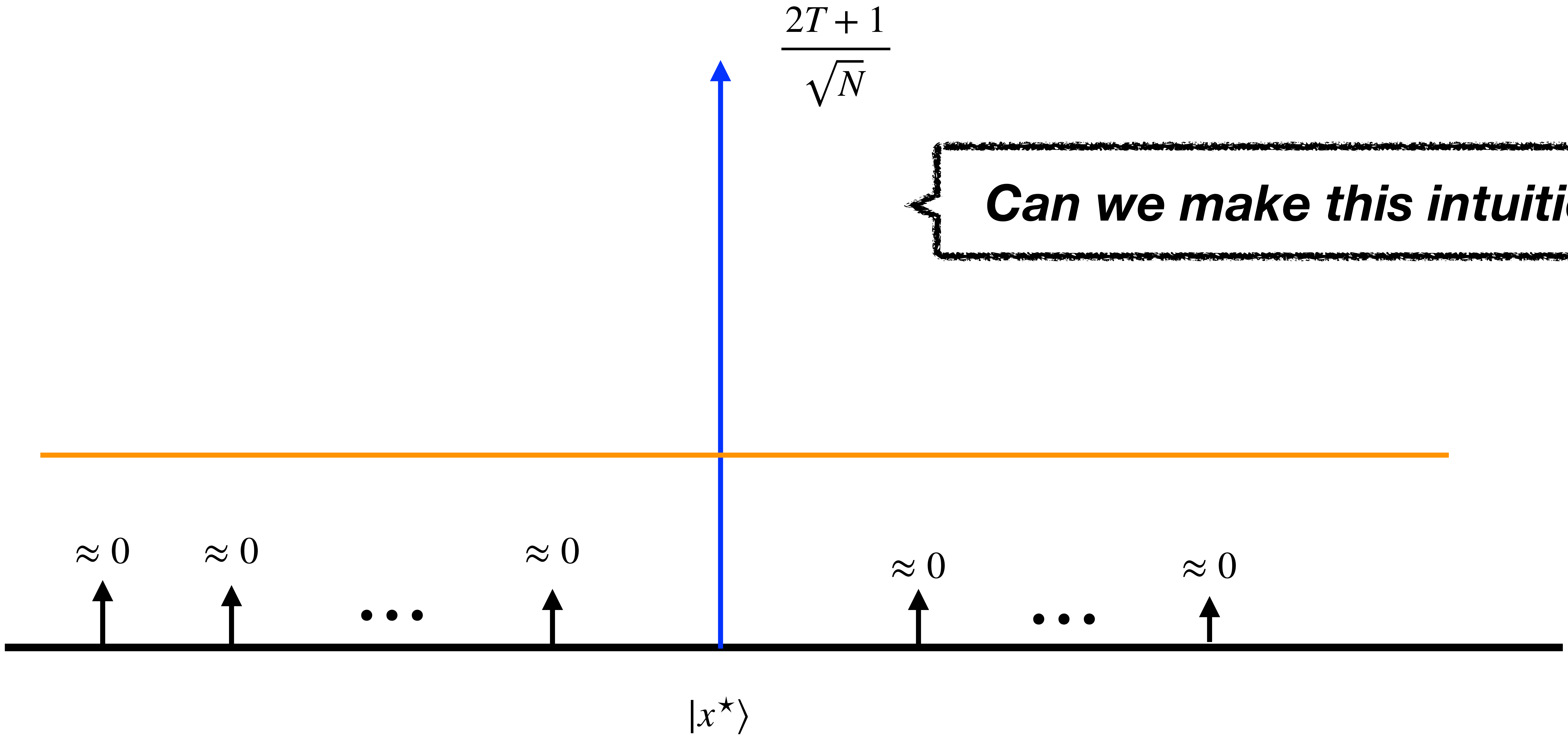

Can we make this intuition rigorous?

After T queries

$$(R_{|u\rangle} O_f)^T |u\rangle$$

Some Definitions

$A = \{x \in \{0,1\}^n \mid f(x) = 1\}$ **Accepting Inputs**

$B = \{x \in \{0,1\}^n \mid f(x) = 0\}$ **Bad Inputs**

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$$|A\rangle = \frac{1}{\sqrt{A}} \sum_{x \in A} |x\rangle \qquad |B\rangle = \frac{1}{\sqrt{B}} \sum_{x \in B} |x\rangle$$

Understanding O_f

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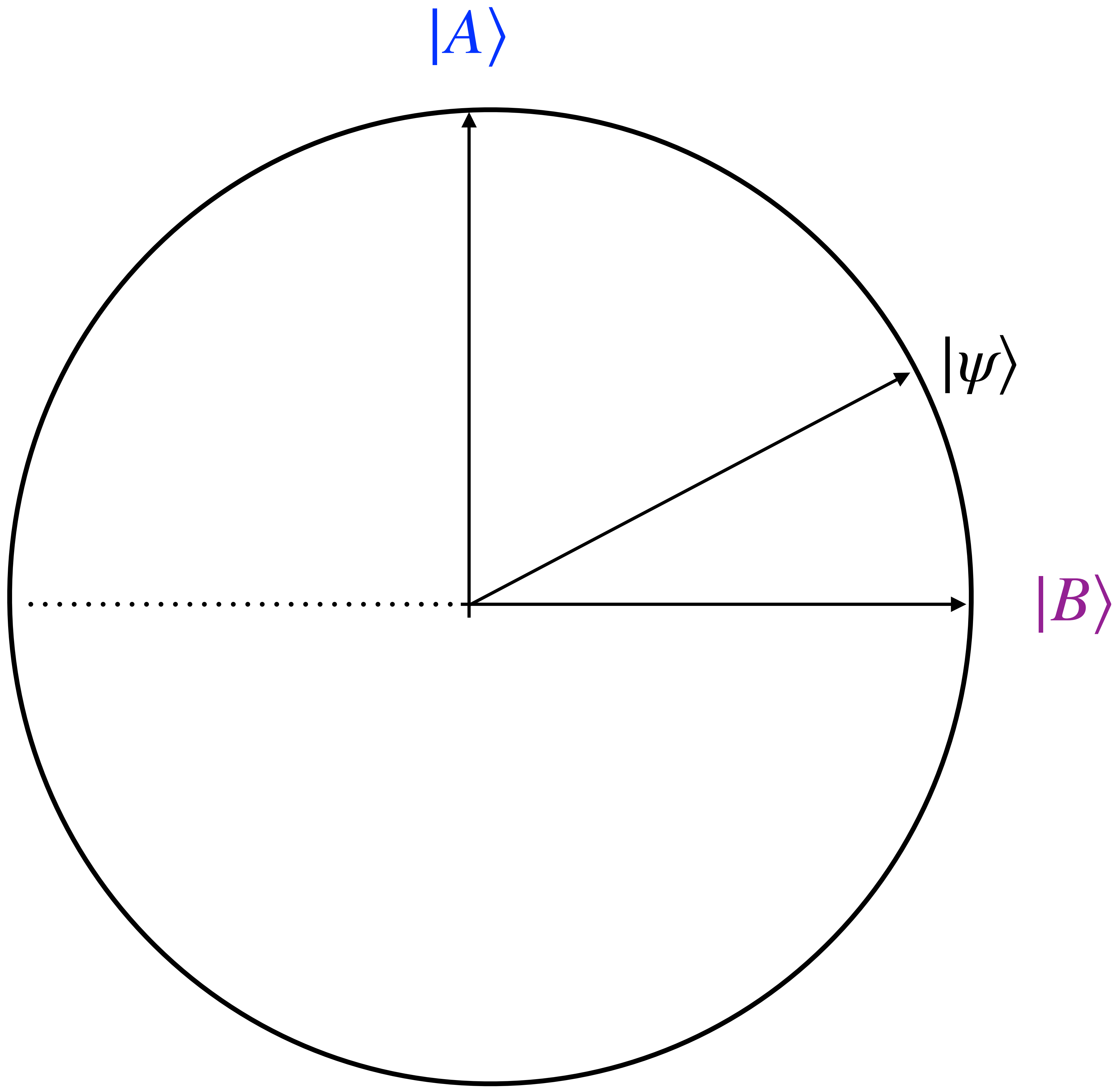

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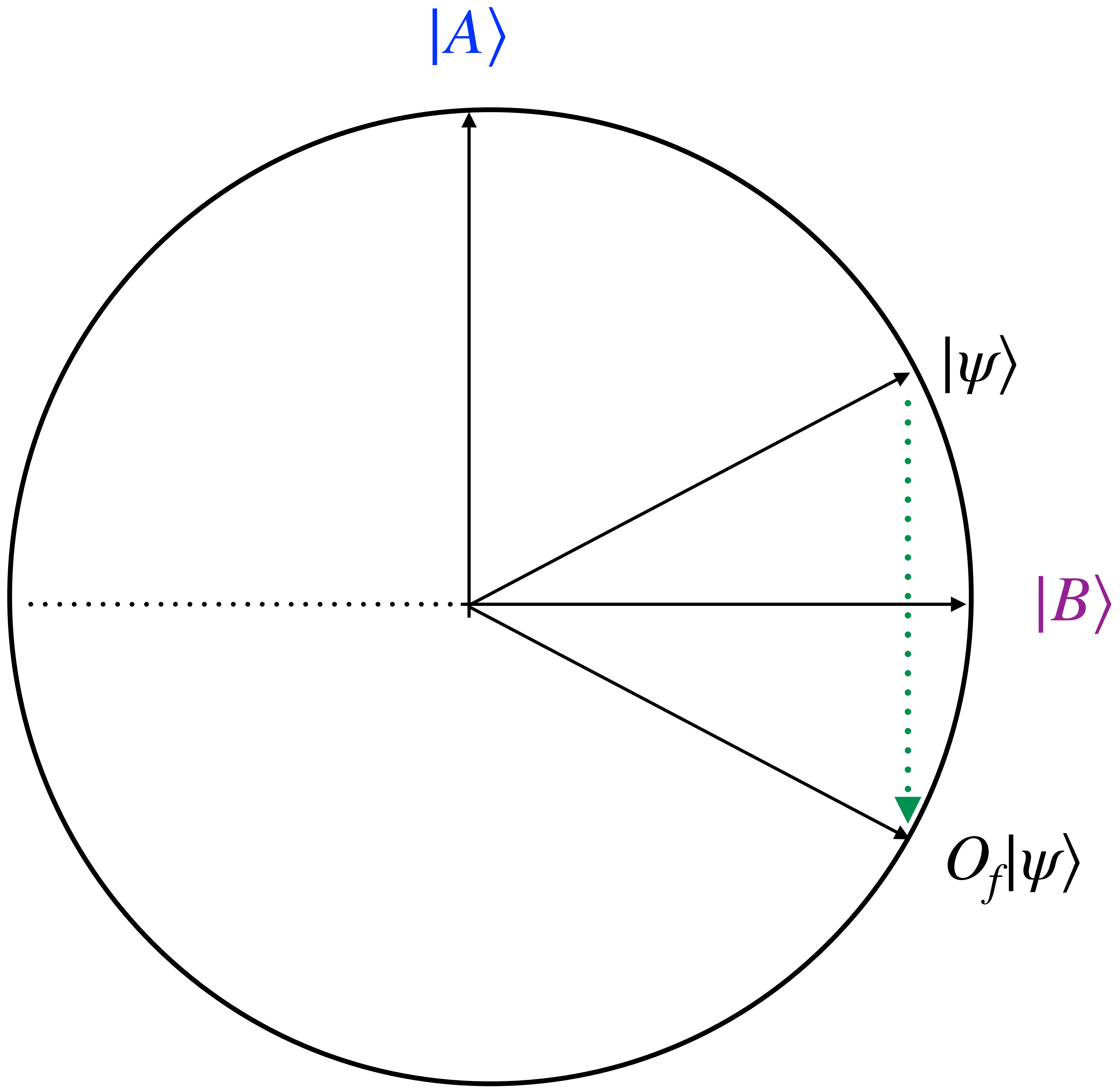

$\langle A \mid B \rangle = 0$

$O_f|A\rangle = -|A\rangle$

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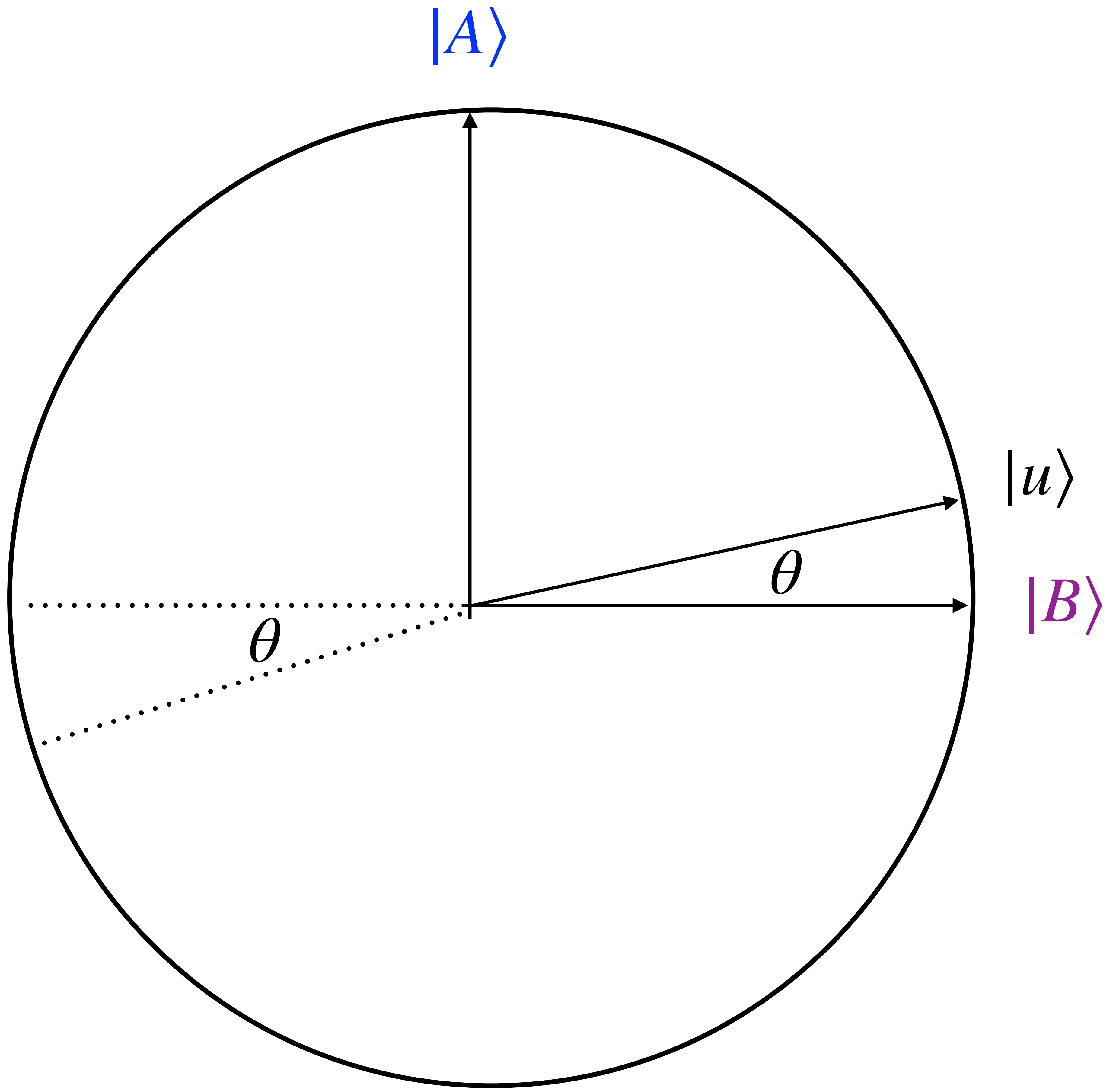
O_f **is a reflection around** $|B\rangle$

Composing two Reflections

$$|u\rangle = \sqrt{\frac{A}{N}}|A\rangle + \sqrt{\frac{B}{N}}|B\rangle$$

$$R_{|u\rangle} = 2|u\rangle\langle u| - I$$

$$R_{|B\rangle} = O_f$$



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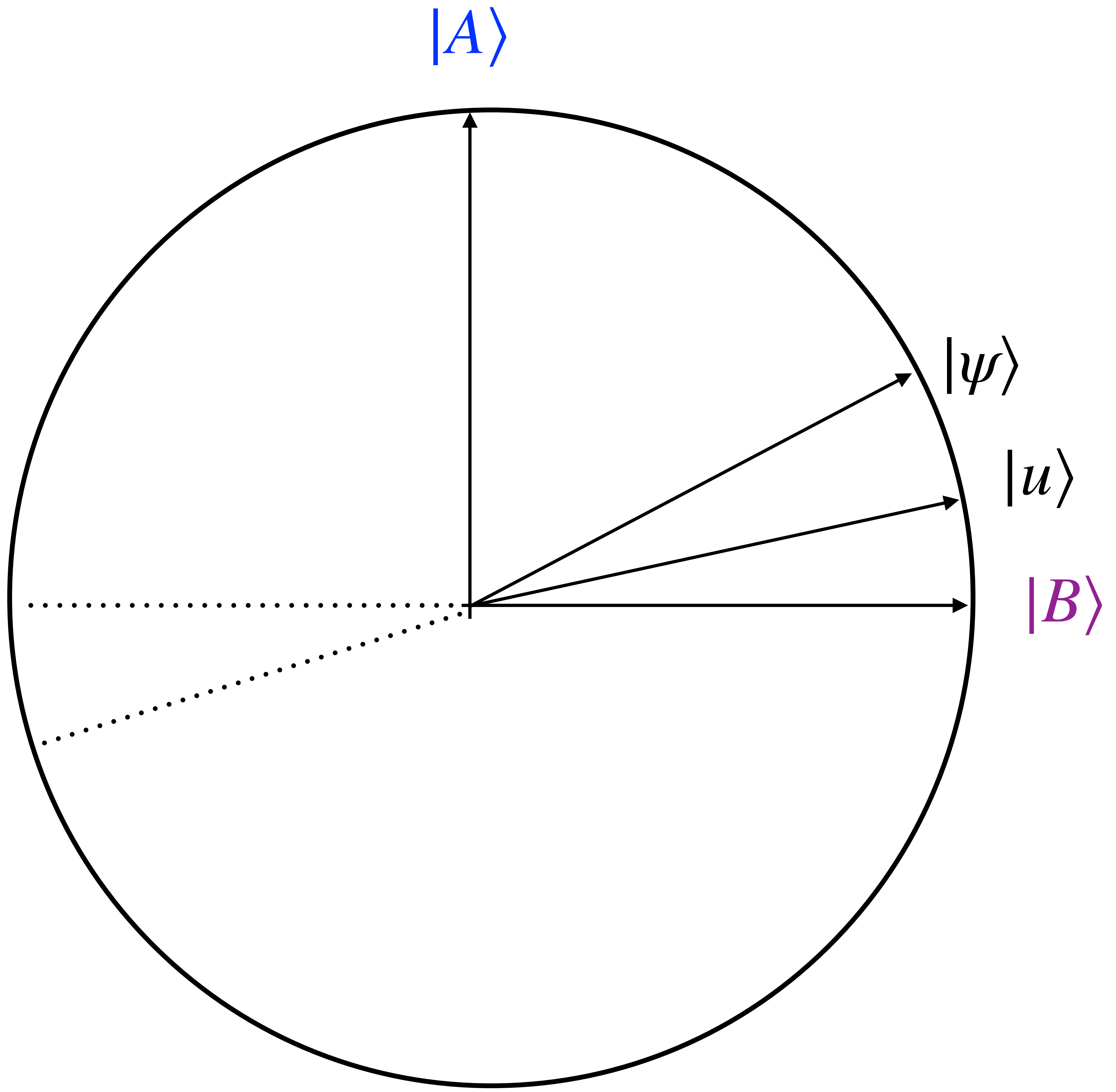

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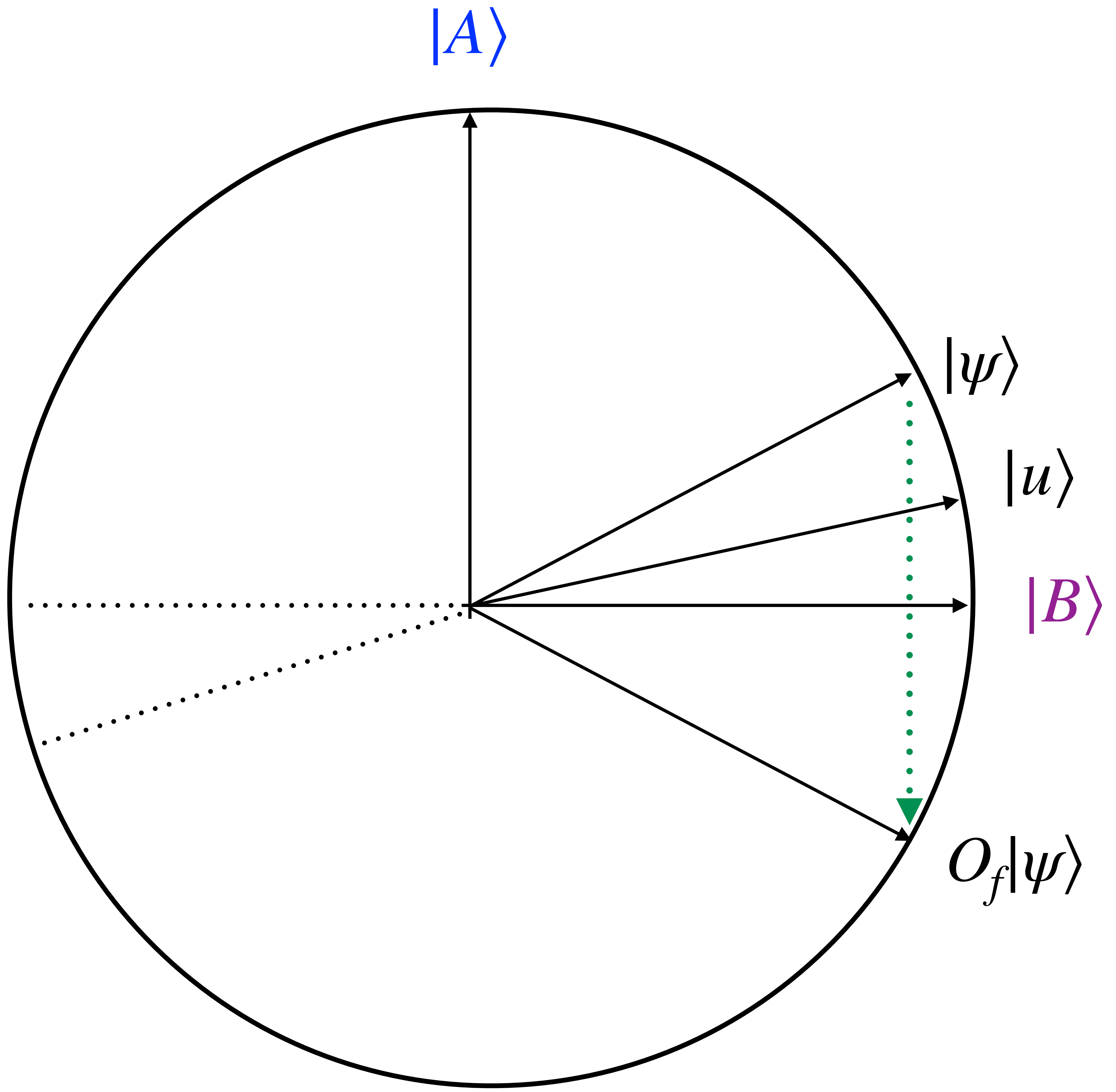

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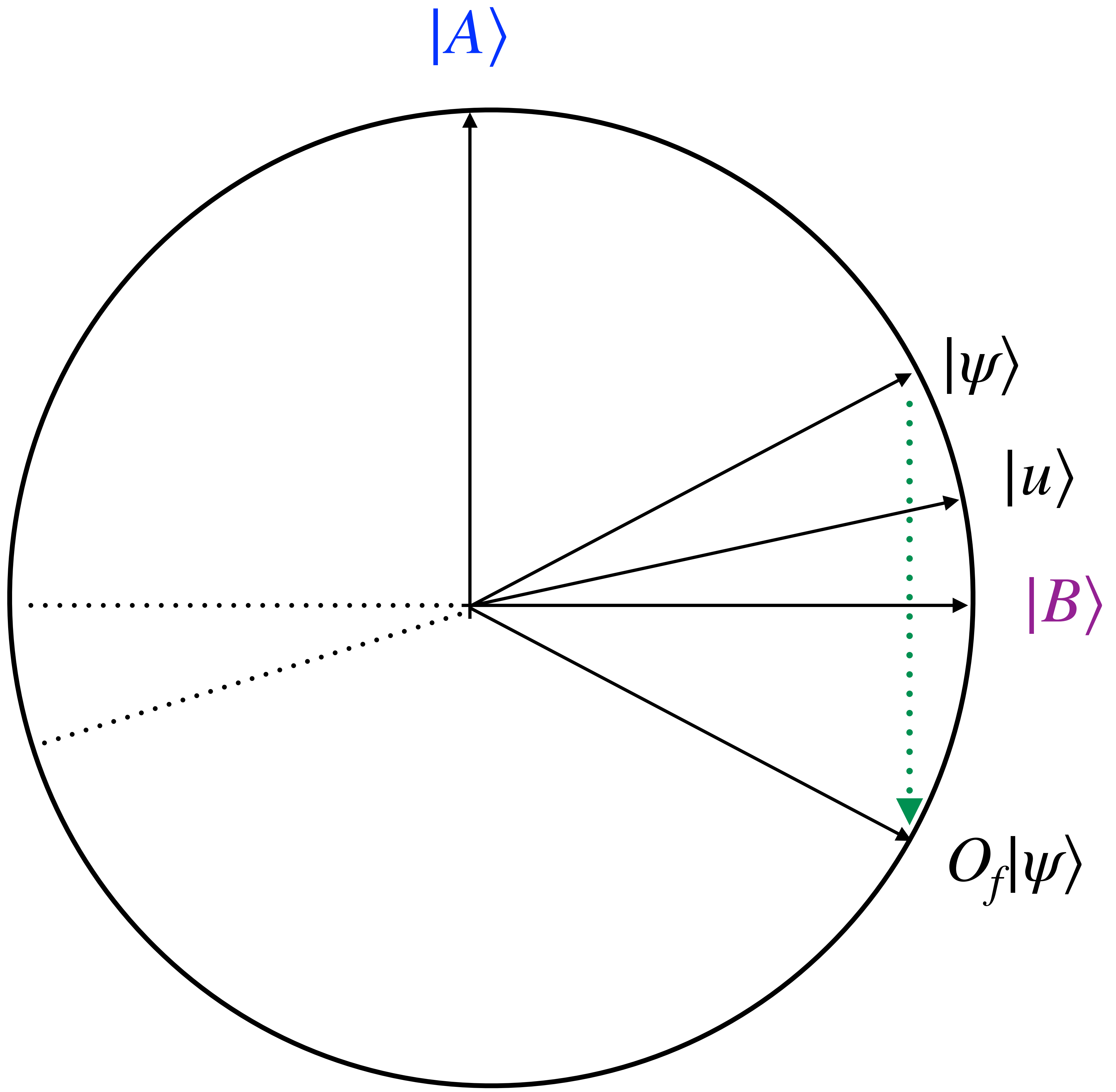

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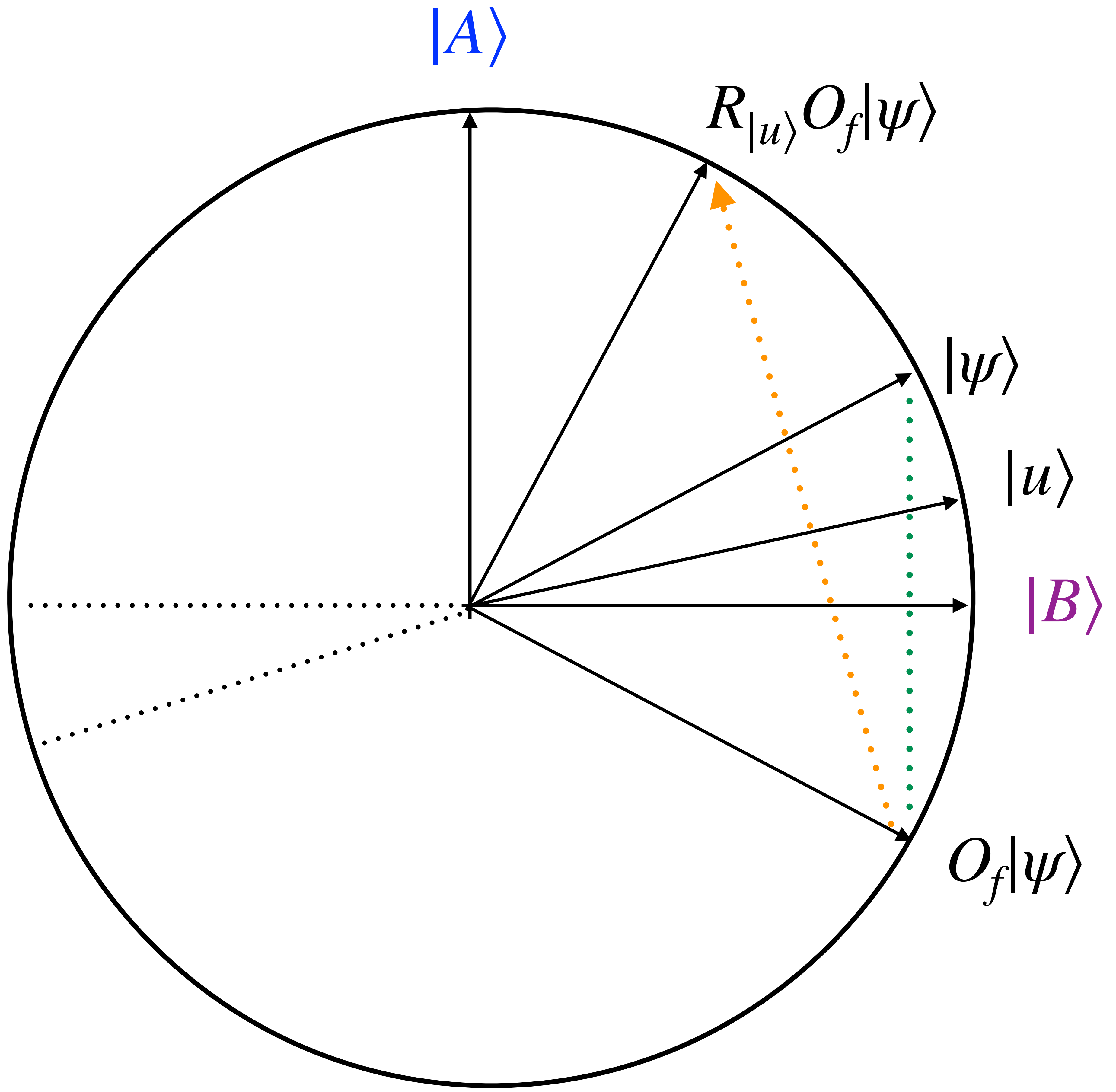

$R_{|u\rangle}$ **is a reflection around** $|u\rangle$

Composing two Reflections gives a Rotation

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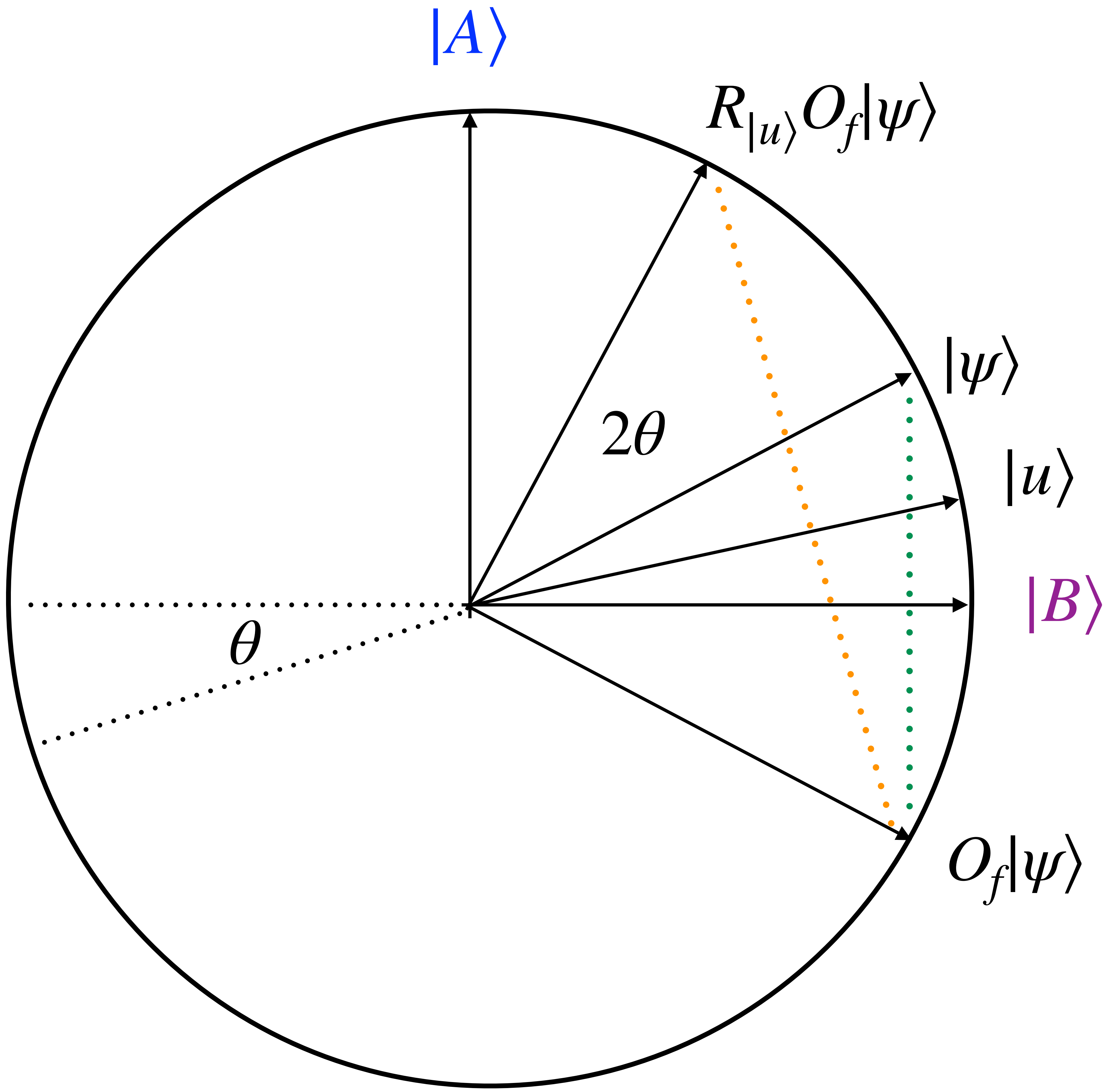
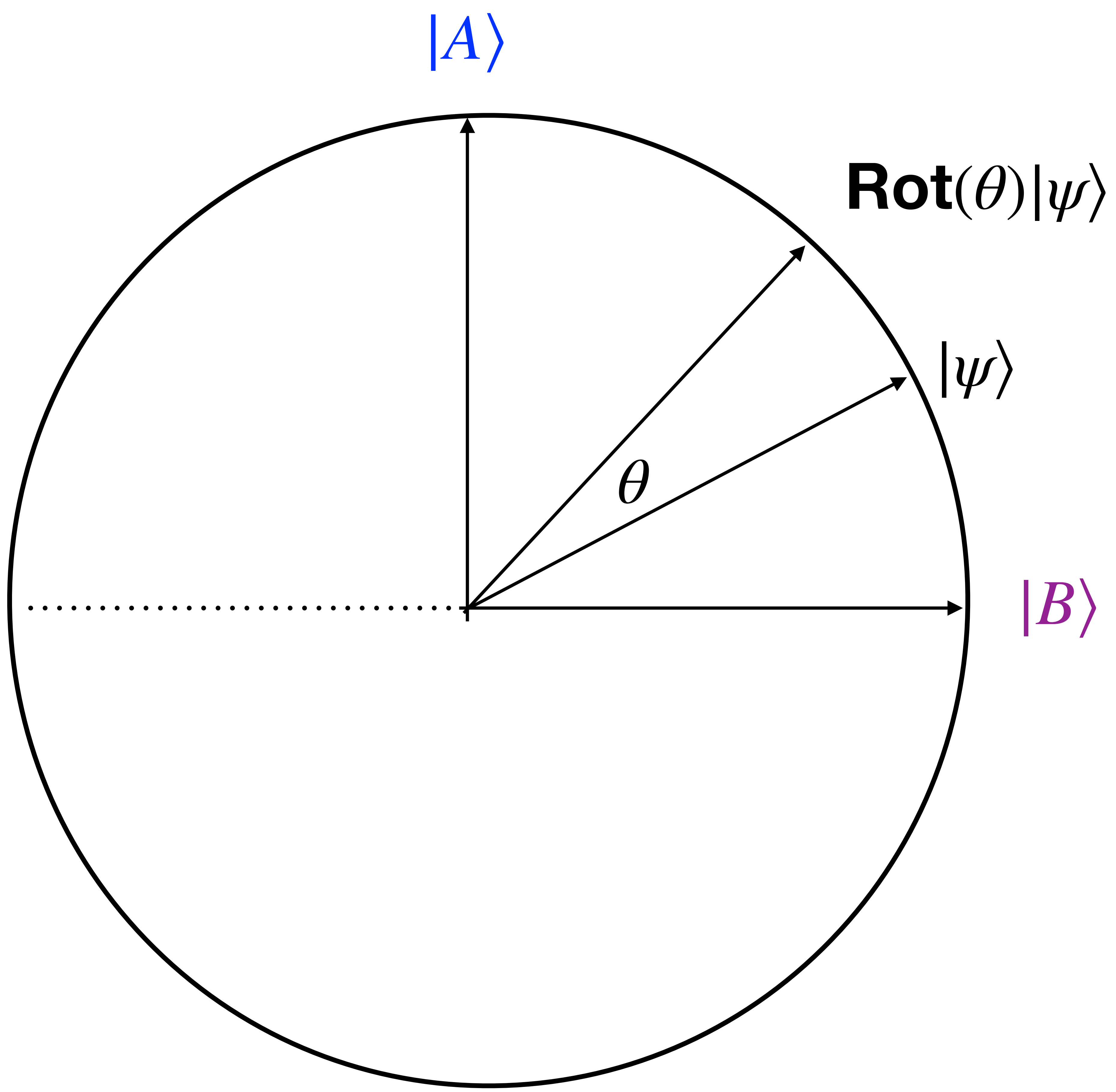


Fact: $R_{|u\rangle}R_{|B\rangle} =$ **Rot**(2θ)

θ **is the angle between** $|u\rangle$ **and** $|B\rangle$

Refresher on Rotation Matrices



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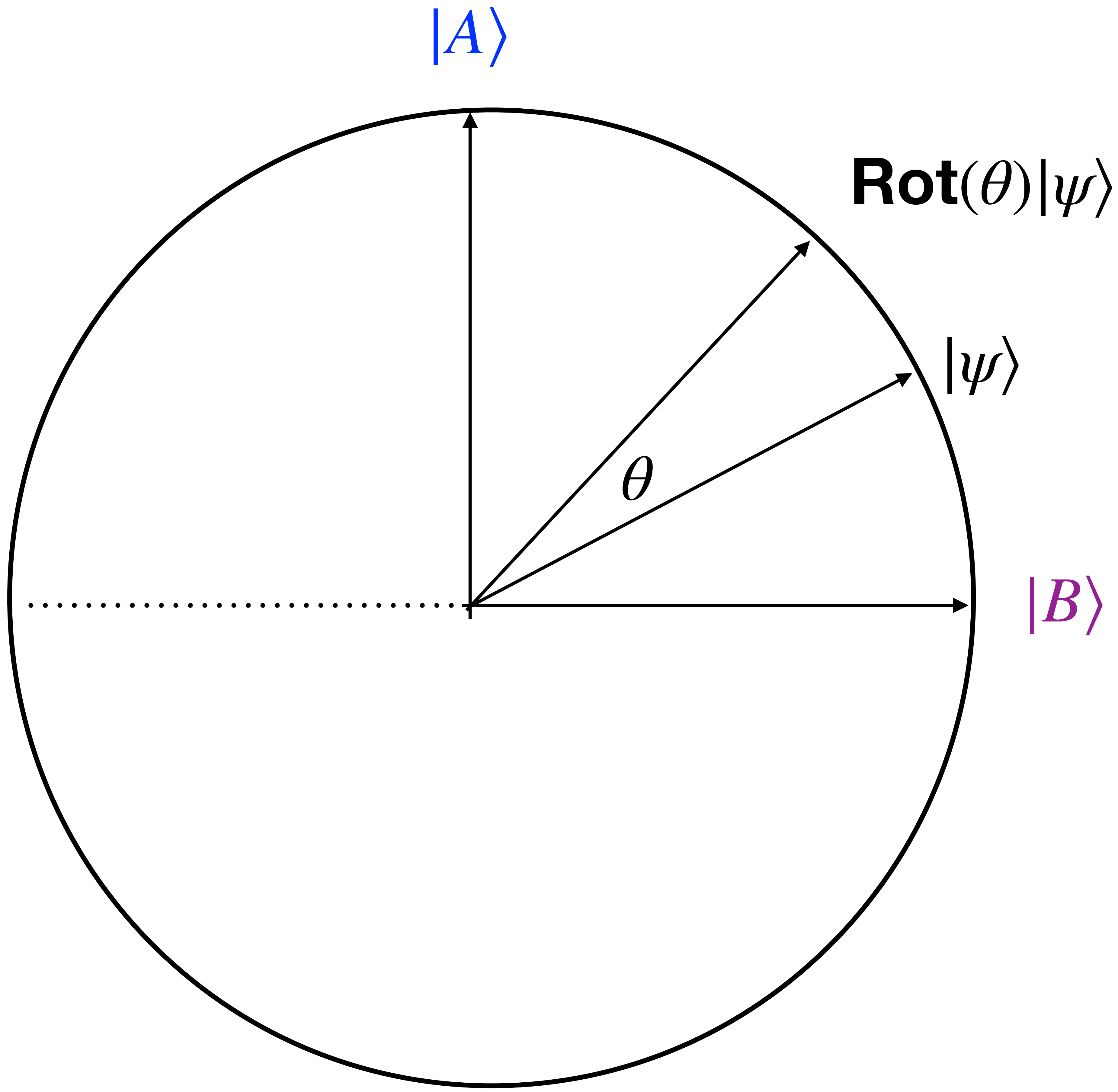

$$\textbf{Rot}(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$\textbf{Rot}(\theta)^k = \begin{pmatrix} \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{pmatrix} = \textbf{Rot}(k\theta)$$

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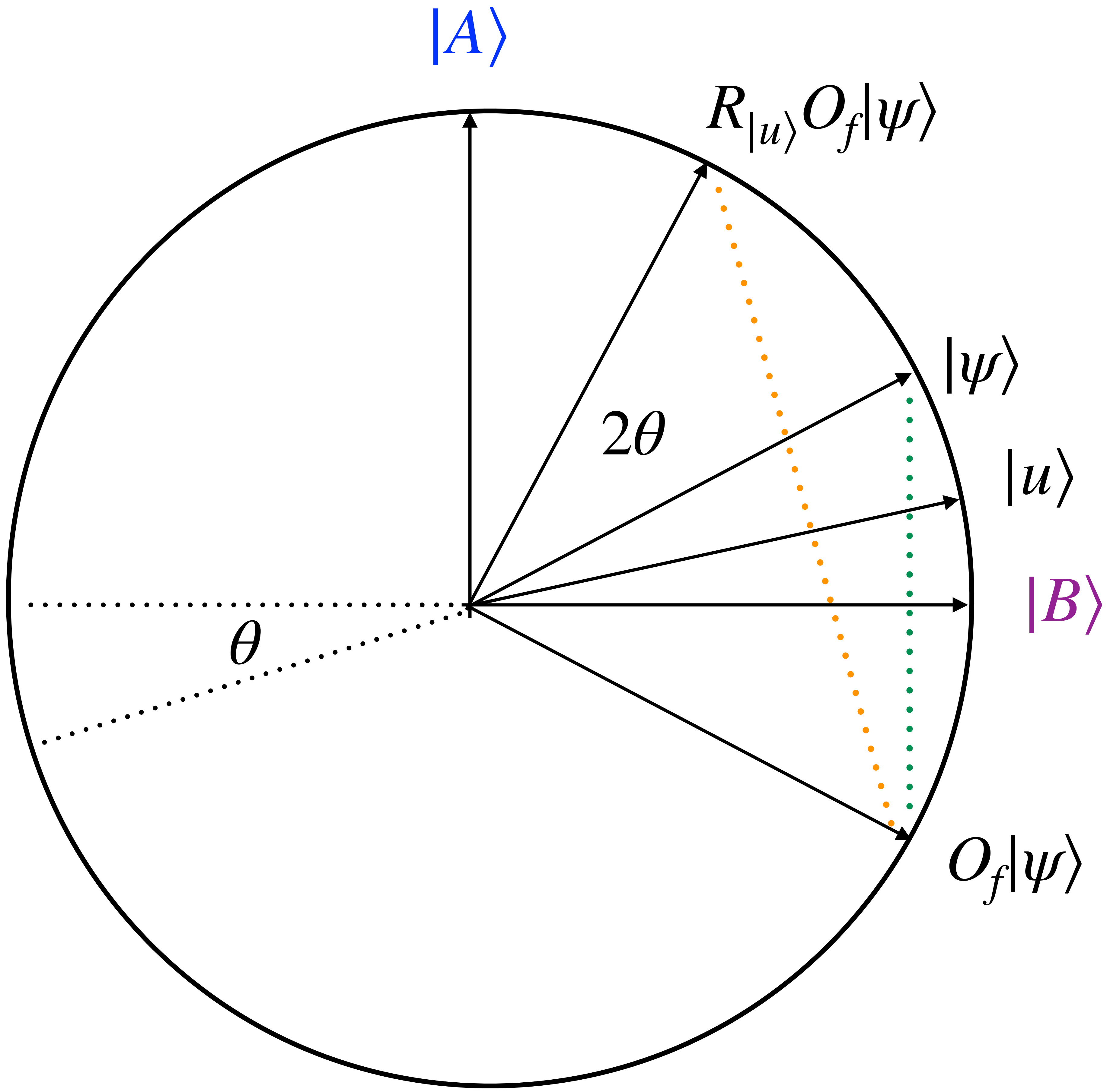

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Analyzing the Number of Queries

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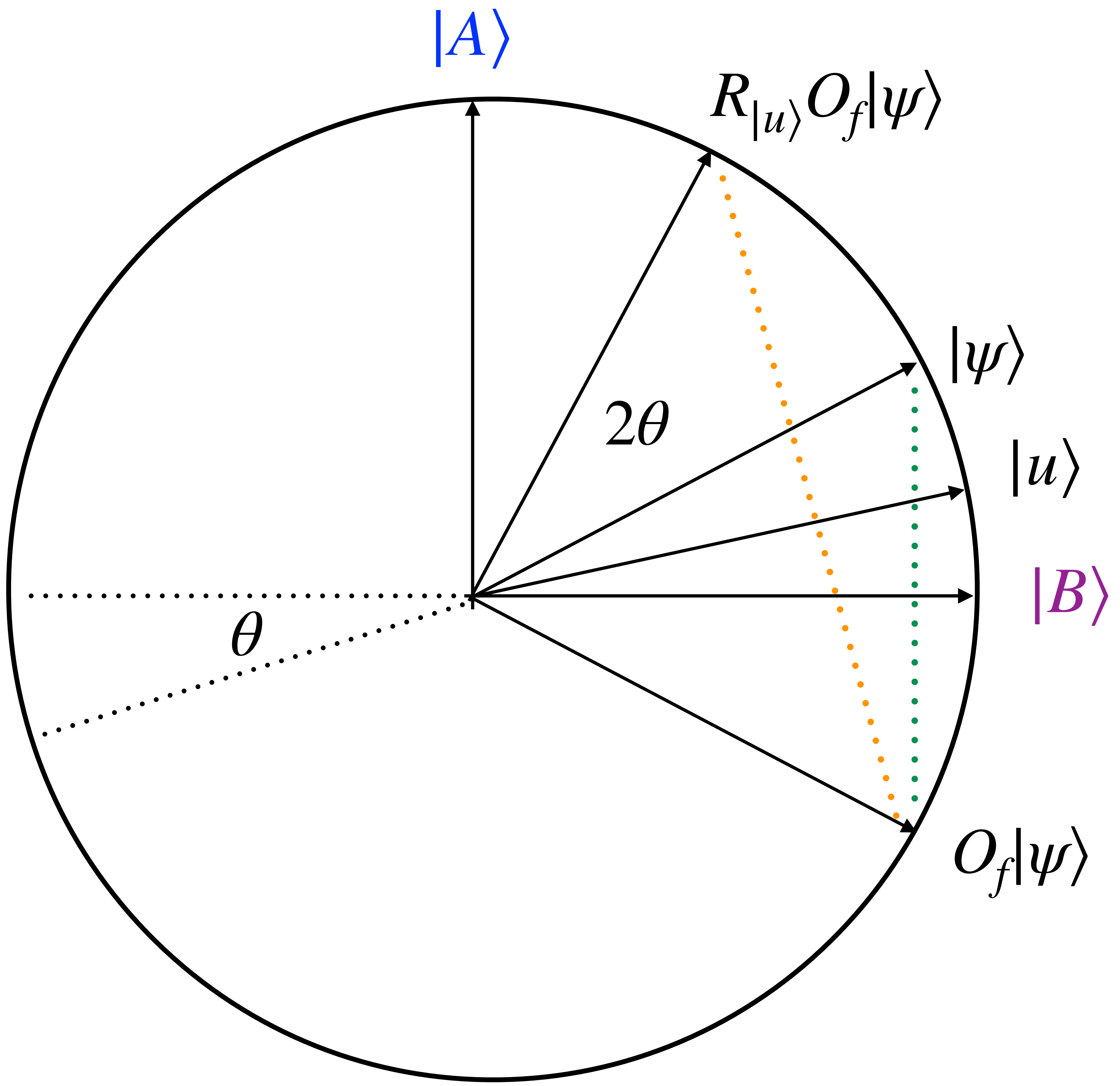

$$(R_{|u\rangle}R_{|B\rangle})^T|u\rangle = \textbf{Rot}(2T\theta)|u\rangle = \sin((2T+1)\theta)|A\rangle + \cos((2T+1)\theta)|B\rangle$$

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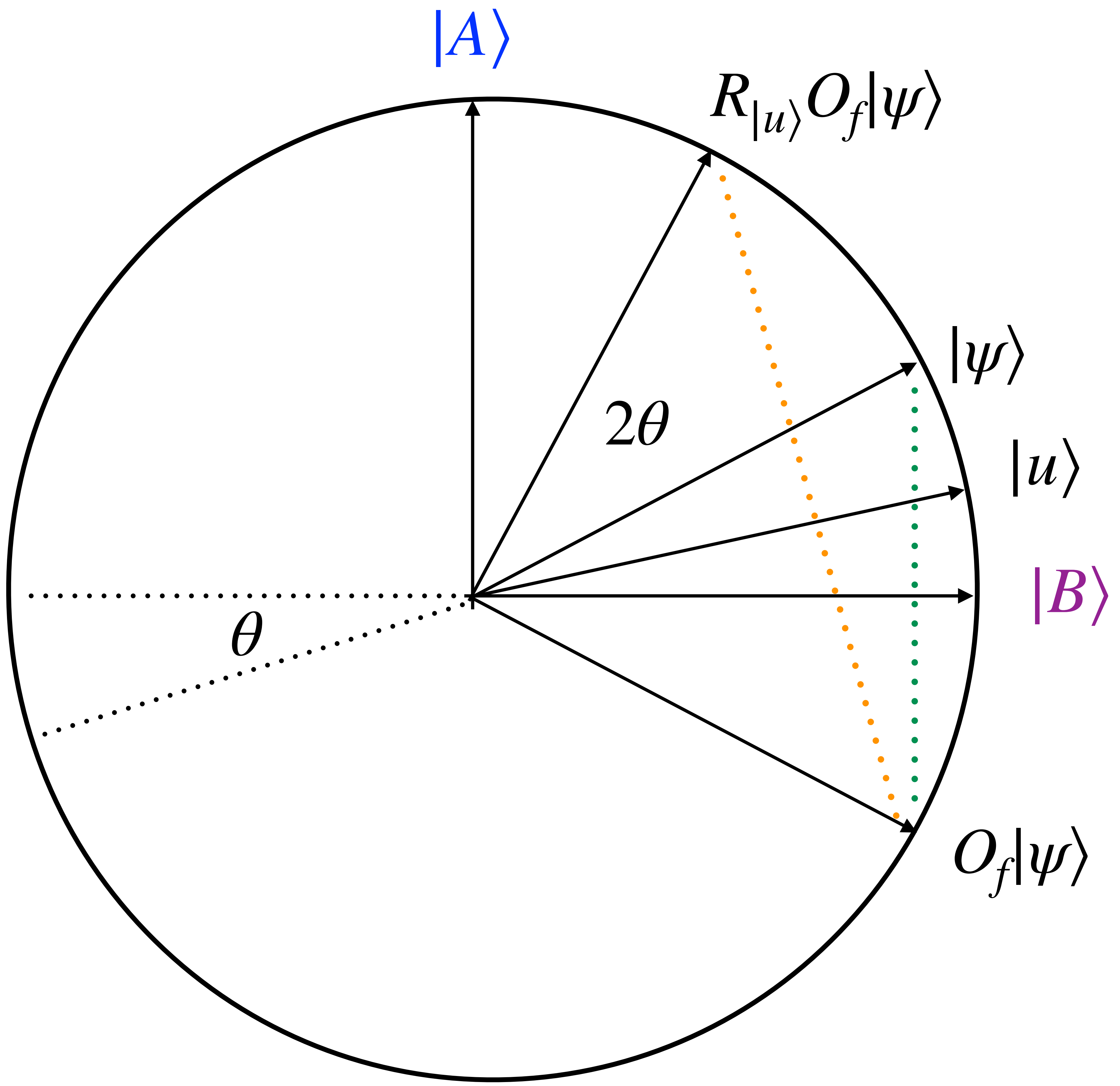

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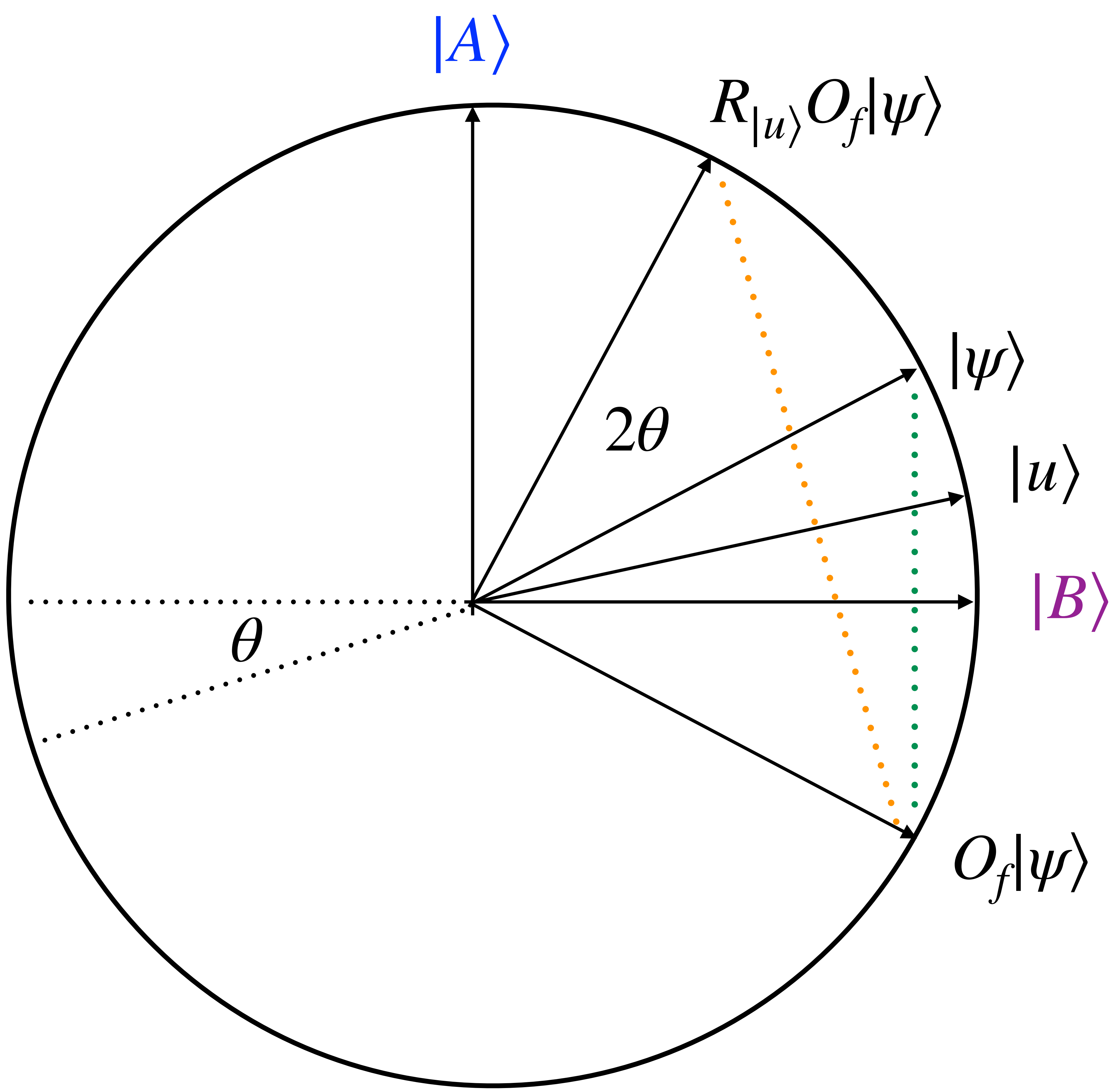

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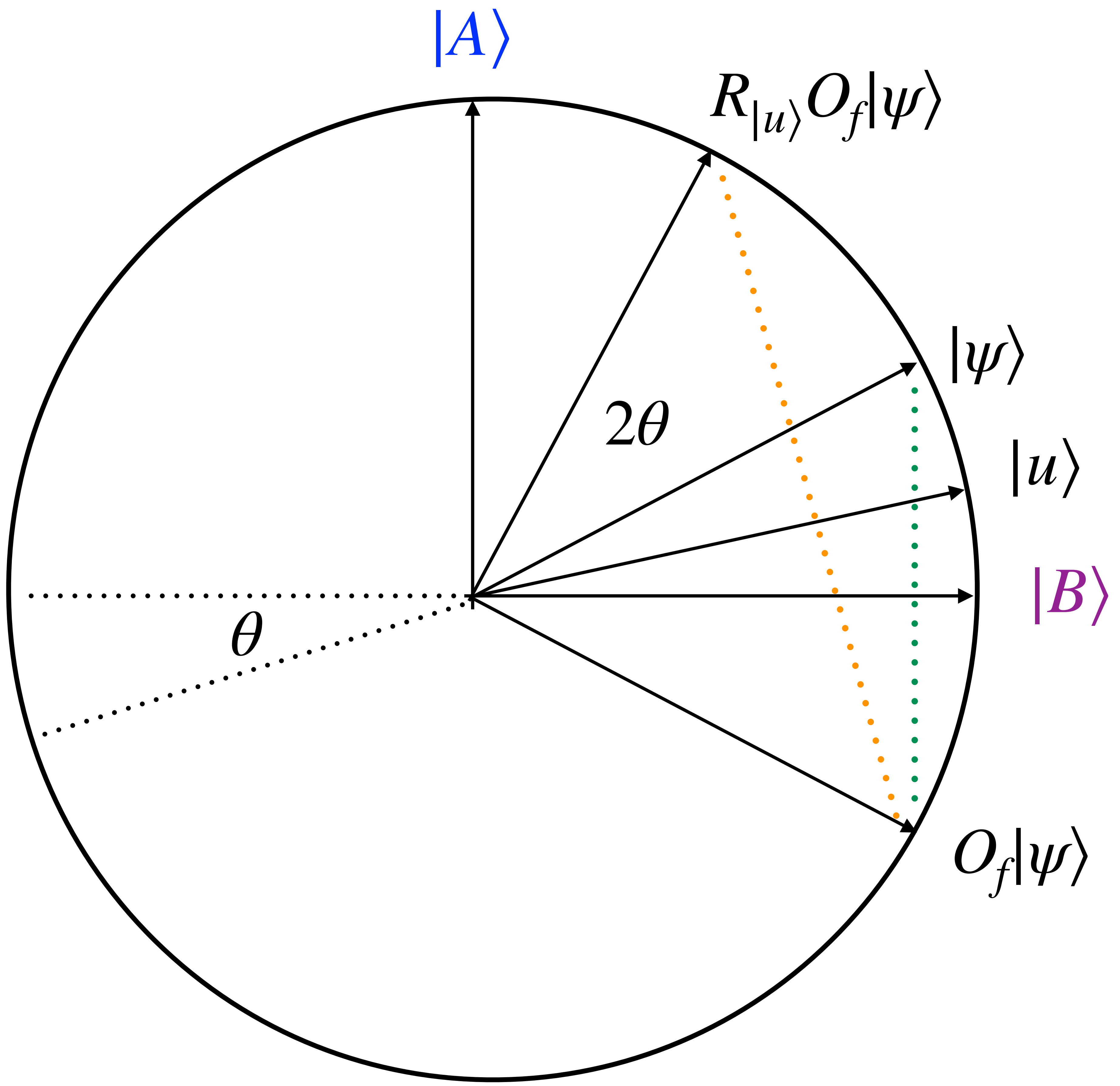

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In Summary: Grover's Algorithm gives a quadratic speedup

<u>**Grover'96**</u>

Unstructured Search can be solved with $O(\sqrt{N})$ quantum queries

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In Summary: Grover's Algorithm gives a quadratic speedup

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Unstructured Search can be solved with $O(\sqrt{N})$ quantum queries

One of the most important quantum algorithms discovered so far

Widely used to obtain many other polynomial quantum speedups

Provably optimum in the black-box query model

Simple Technicalities to Handle

$$|u\rangle = \sqrt{\frac{A}{N}}|A\rangle + \sqrt{\frac{B}{N}}|B\rangle = \sin(\theta)|A\rangle + \cos(\theta)|B\rangle$$

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We (implicitly) assumed that $|A| \neq 0$ (the case $|A| = 0$ is easy to analyze, do you see this?)

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We (implicitly) assumed that $|A| \neq 0$ (the case $|A| = 0$ is easy to analyze, do you see this?)

Can we implement the reflection $R_{|u\rangle}$ efficiently?

We do not know $|A|$ in advance, can think of strategies to deal with this?

Number of Queries in Grover's Algorithm

Can we do better (fewer queries) when $|A|$ is large?

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<u>Grover's Search</u>

Unstructured Search can be solved with $O\left(\sqrt{\frac{N}{A}}\right)$ ***quantum queries if*** $|A| \neq 0$

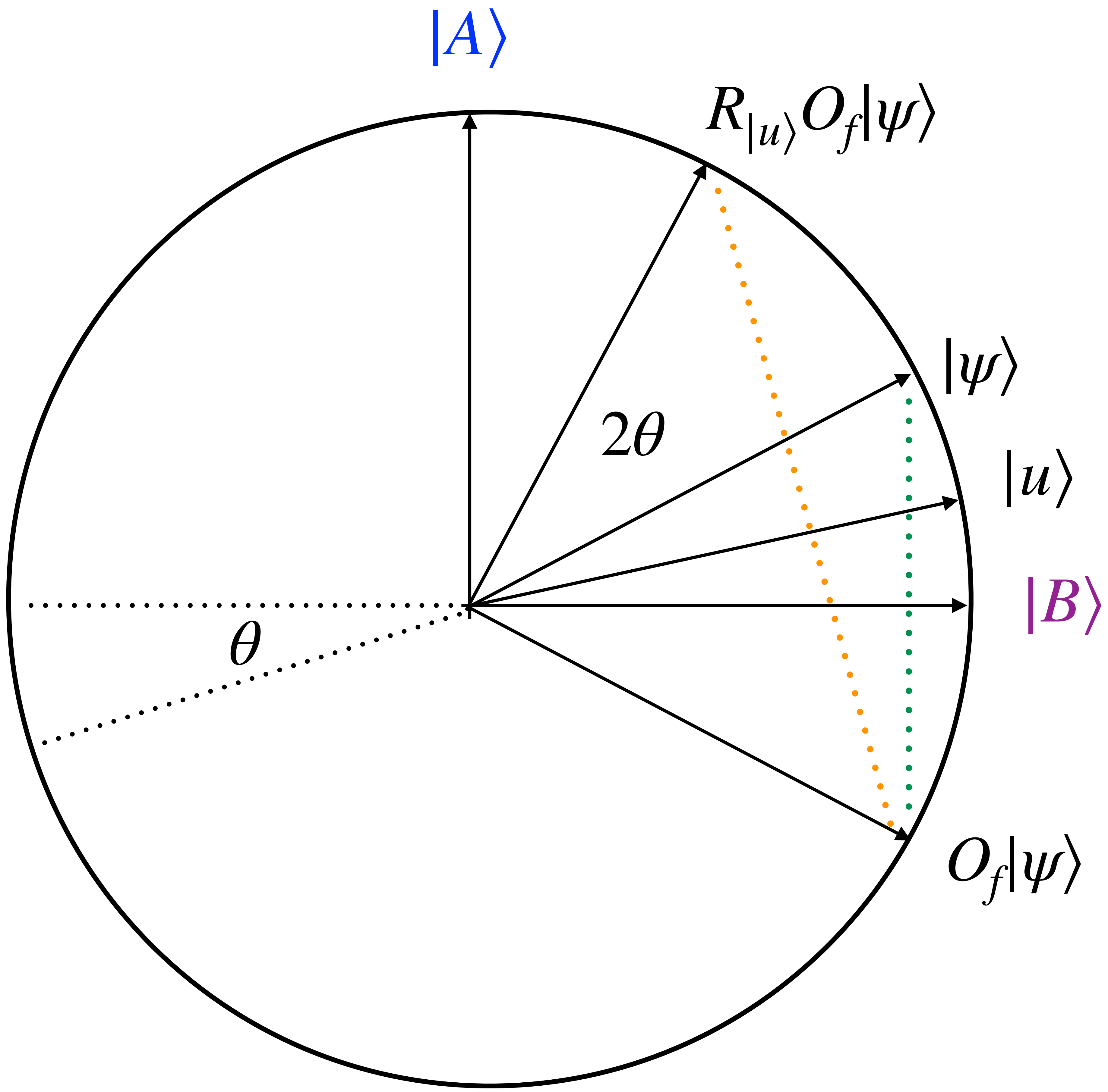
Completing the Proofs…

Proving the Composition Fact

$|u\rangle = \sqrt{\frac{A}{N}}|A\rangle + \sqrt{\frac{B}{N}}|B\rangle$

$R_{|u\rangle} = 2|u\rangle\langle u| - I$

$R_{|B\rangle} = O_f$



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θ is the angle between $|u\rangle$ and $|B\rangle$

$$R_{|u\rangle}R_{|B\rangle}|A\rangle = -R_{|u\rangle}|A\rangle = -(2|u\rangle\langle u| - I)|A\rangle$$

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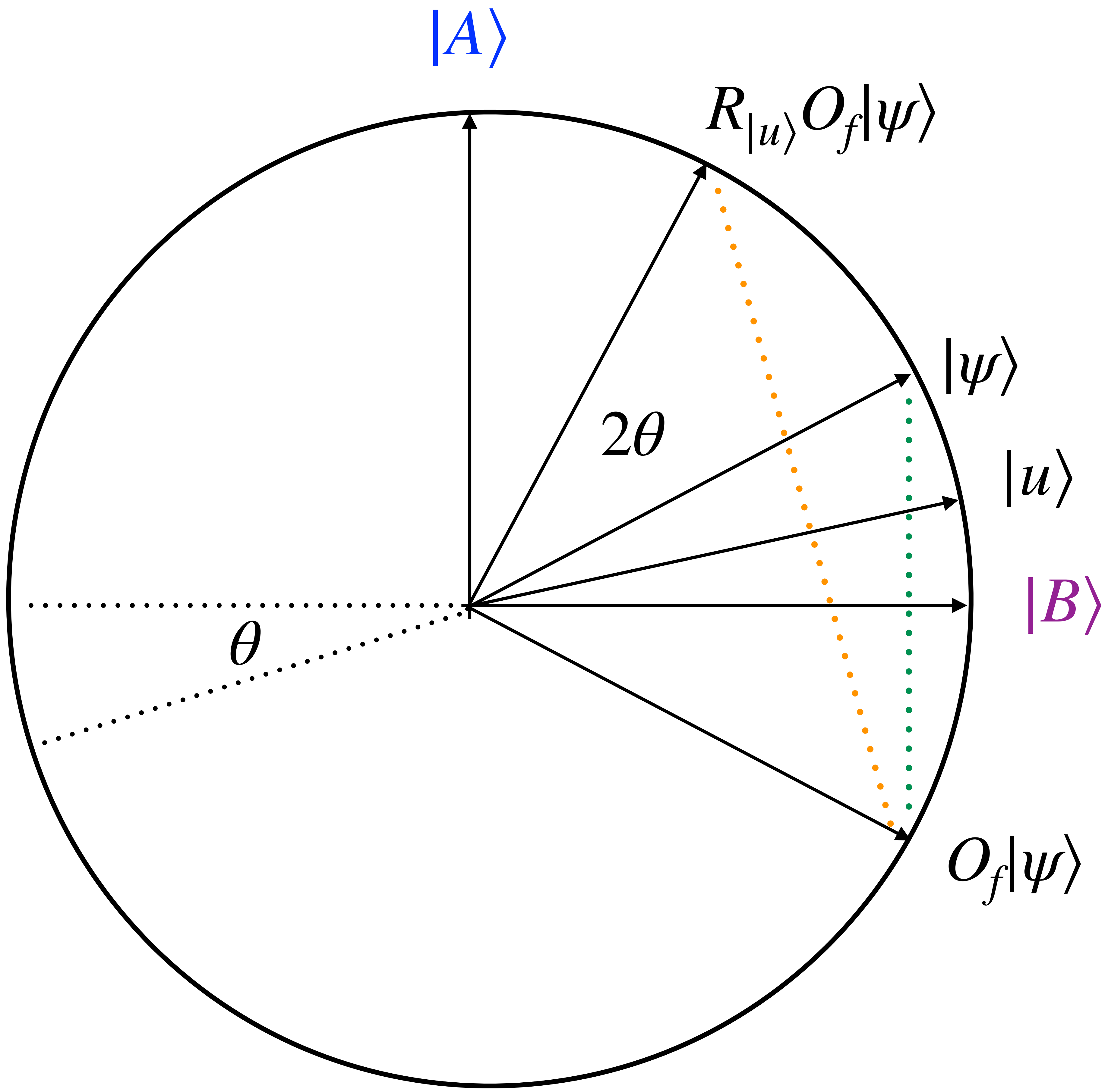

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$$R_{|u\rangle}R_{|B\rangle}|A\rangle = -R_{|u\rangle}|A\rangle = -(2|u\rangle\langle u| - I)|A\rangle$$

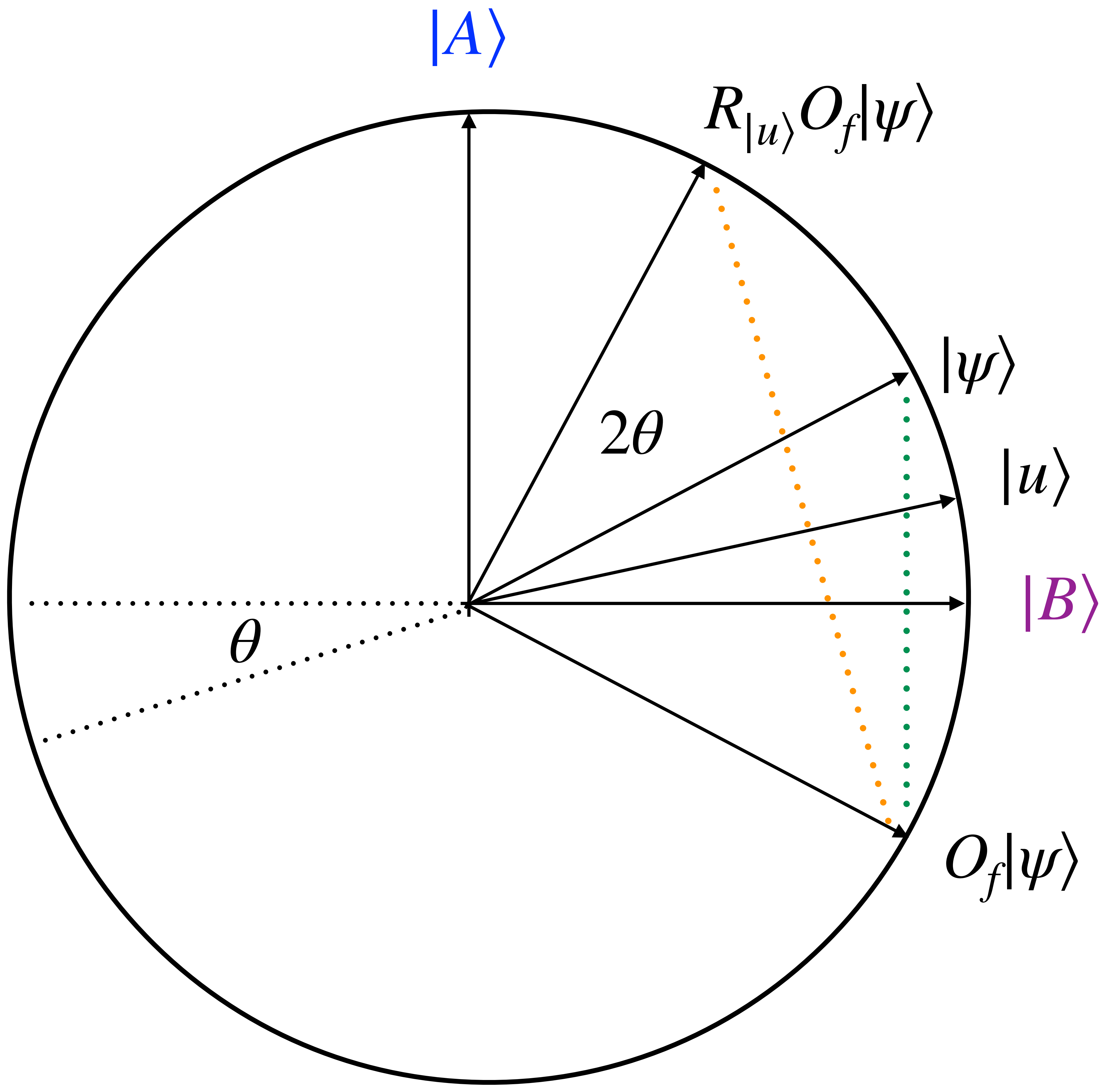
$$= -2\sqrt{\frac{A}{N}}|u\rangle + |A\rangle$$

Proving the Composition Fact

$$R_{|u\rangle} = 2|u\rangle\langle u| - I$$

$$R_{|B\rangle} = O_f$$

$$|u\rangle = \sqrt{\frac{A}{N}}|A\rangle + \sqrt{\frac{B}{N}}|B\rangle$$



Fact: $R_{|u\rangle}R_{|B\rangle} = \textbf{Rot}(2\theta)$

θ is the angle between $|u\rangle$ and $|B\rangle$

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$$= -2\sqrt{\frac{A}{N}}|u\rangle + |A\rangle$$

$$= \frac{N-2A}{N}|A\rangle - \frac{2\sqrt{AB}}{N}|B\rangle$$

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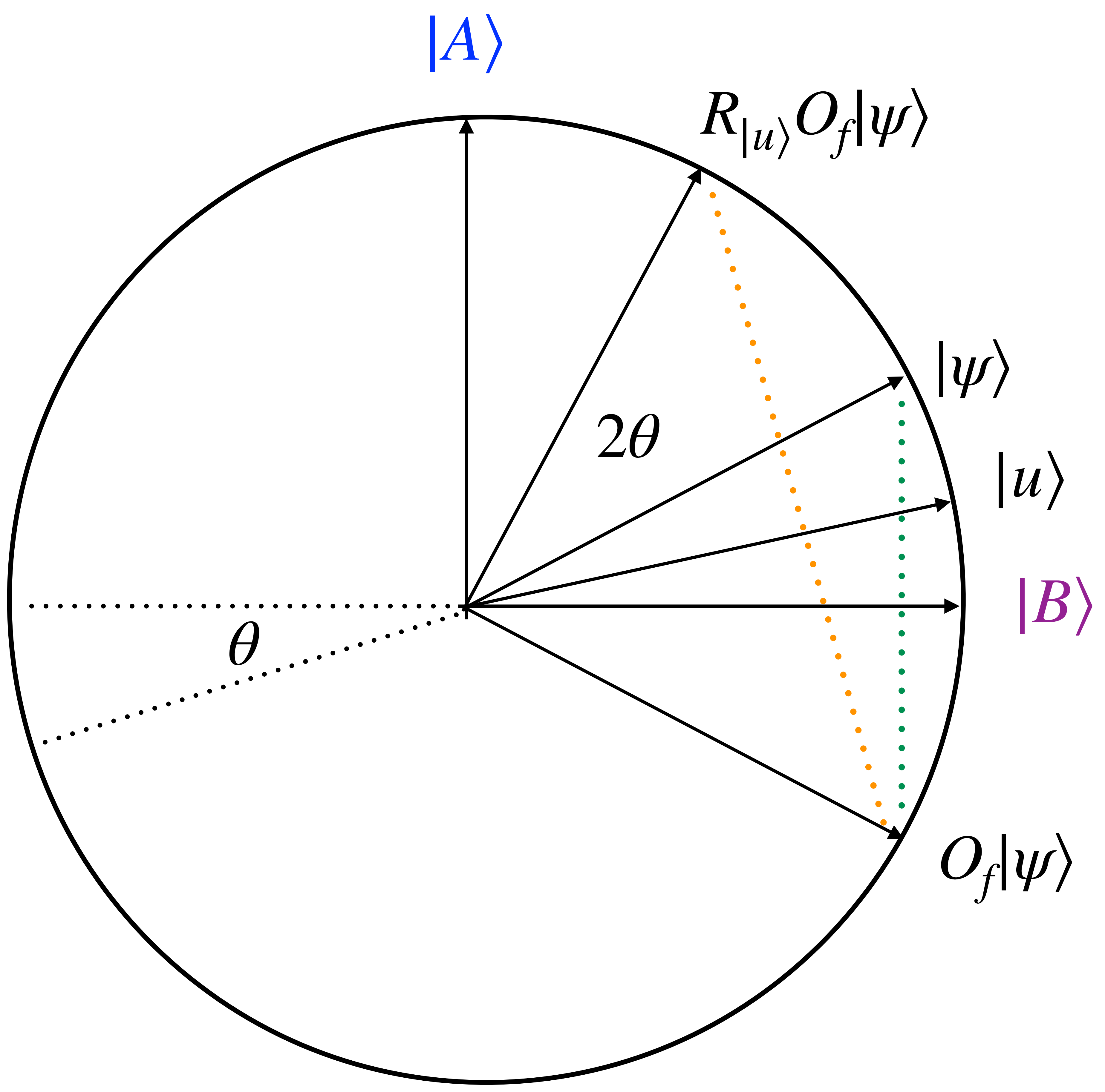

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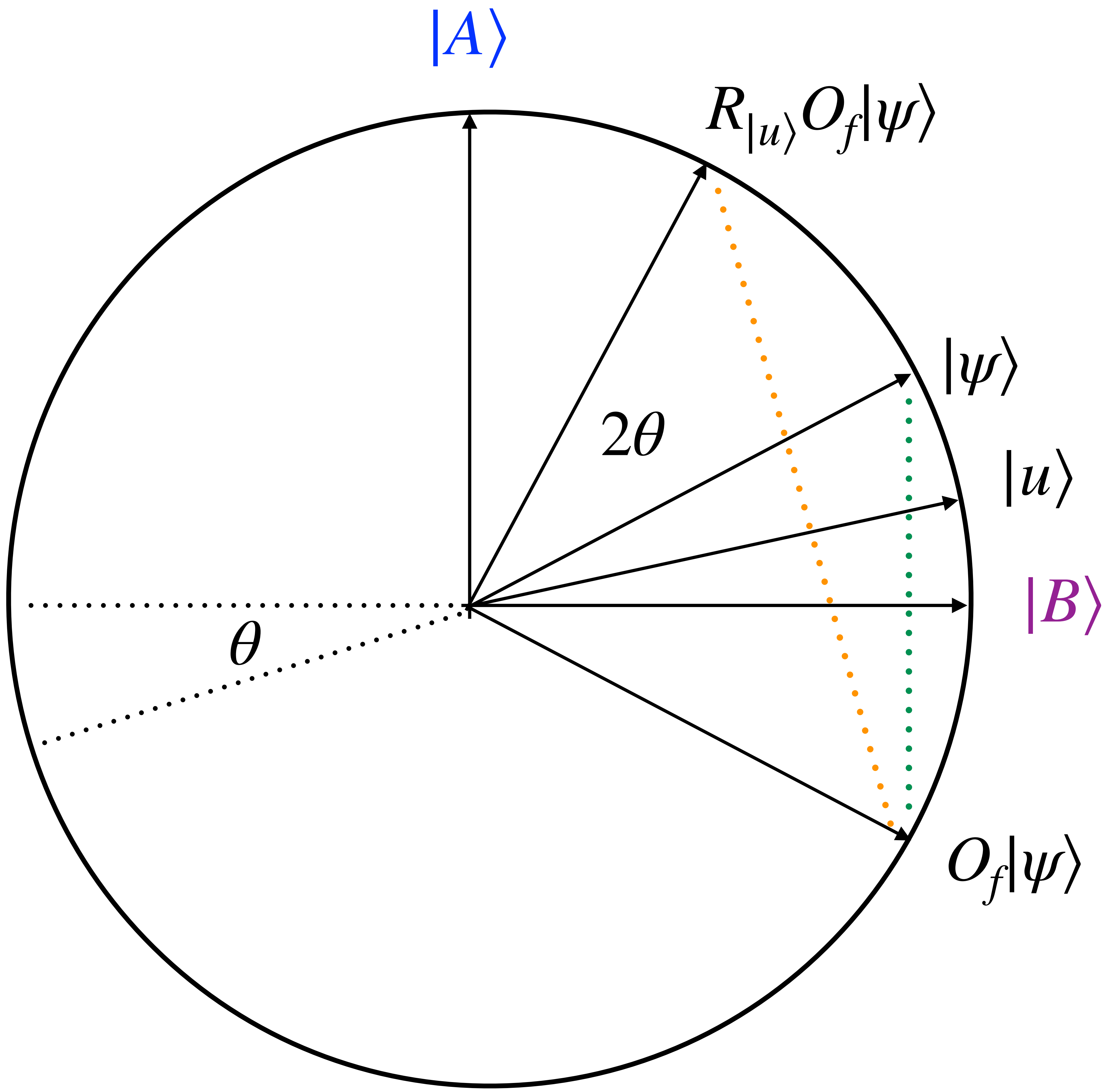

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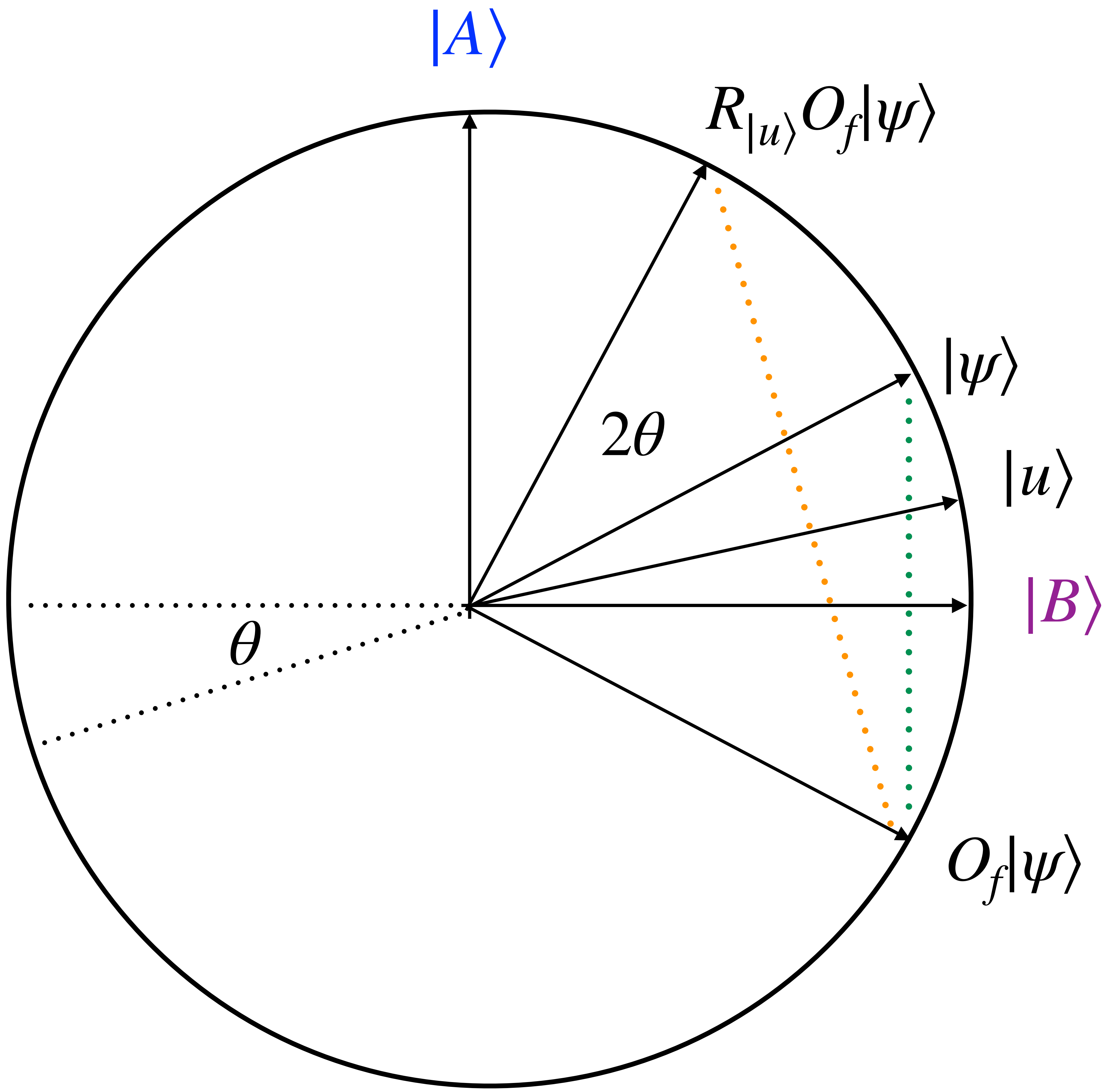

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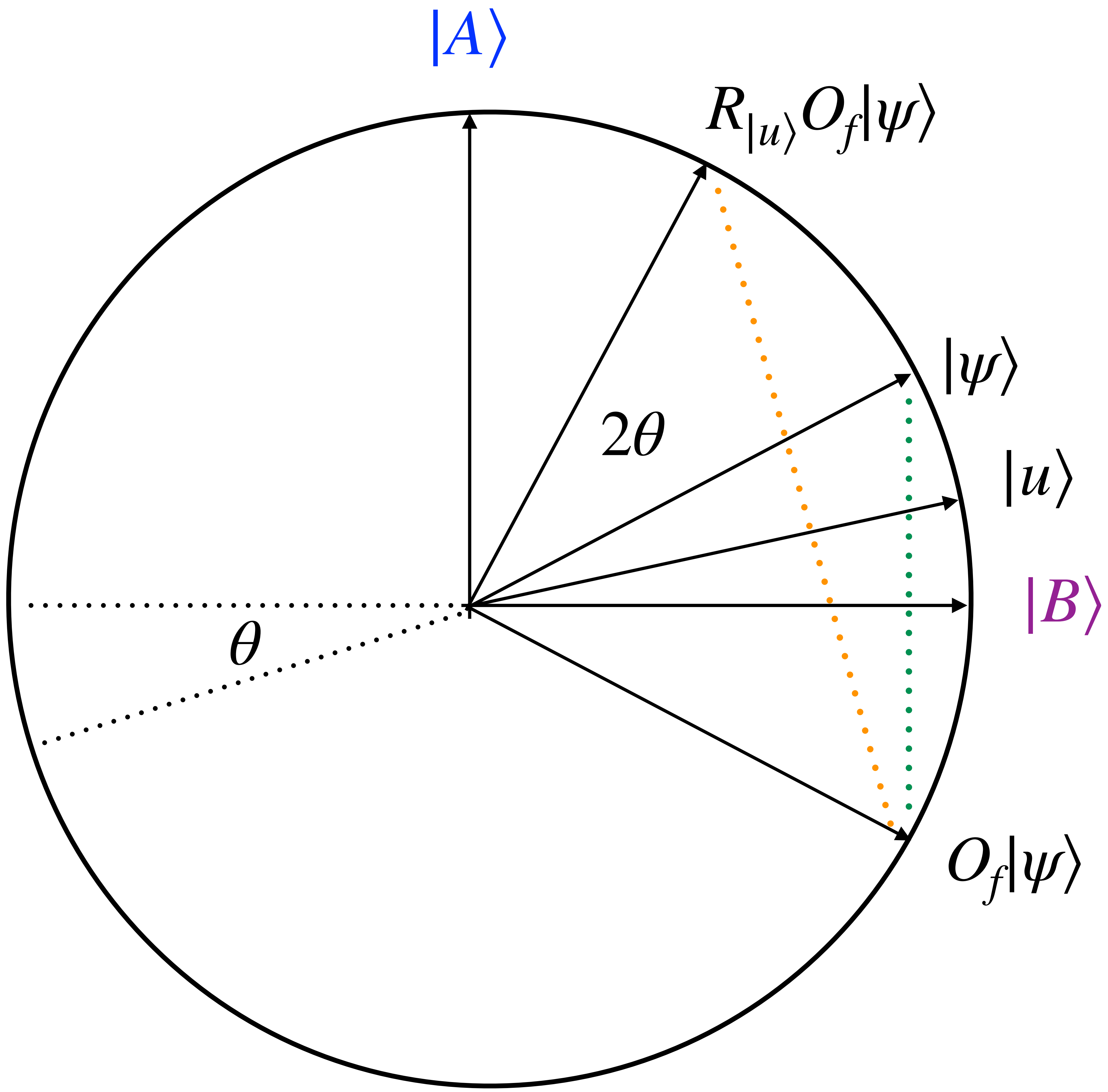

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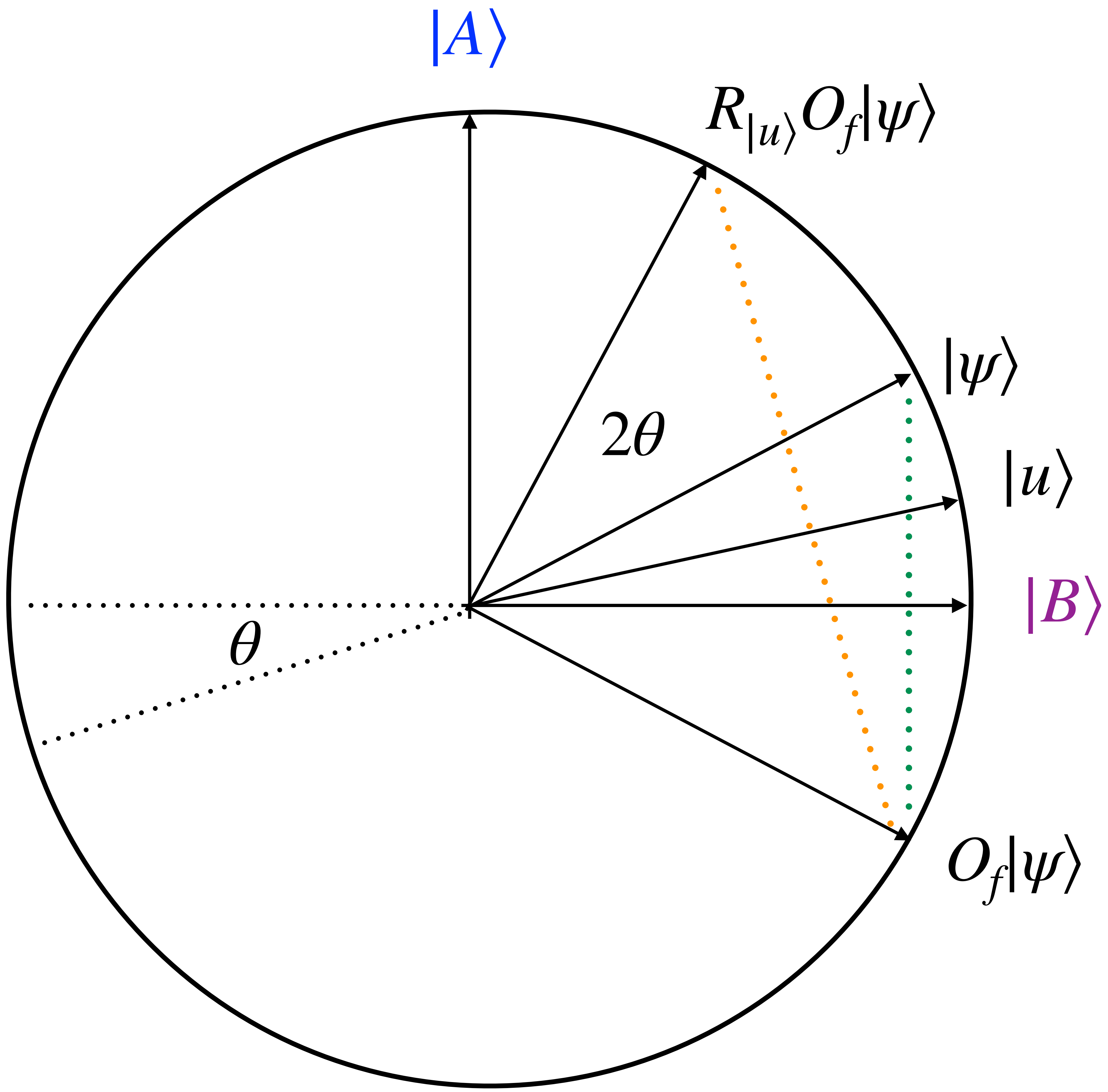

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Can you implement other reflections?

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More Questions?