Discovering the Foundations of Theoretical Computer Science

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$\boxed{\text{Under heavy construction}}$

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Inspired by Babai's approach and the hungarian school

Van Gogh prize

a symbolic prize for any student in the course that solves an important <u>open problem</u>

(like van Gogh you are not going to receive anything other than have done something amazing)

Warning: these open problems can be challenging

Spectral Lens

Let $G=(V,E)$ be a d-regular graph on n vertices.

Let A be its adjacency matrix, i.e.,

$$A \in \mathbb{R}^{n\times n}, \qquad A_{u,v} = \mathbb{1}_{[\{u,v\}\in E]}.$$

Study the spectral theorem.

Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of A with corresponding orthonormal eigenvectors $\varphi_1, \dots, \varphi_n \in \mathbb{R}^n$.

$$(A\varphi_i = \lambda_i \varphi_i)$$

1] Prove that d is an eigenvalue of A.

Def $\langle x, x \rangle = \sum_{i=1}^{n} \bar{x}_i x_i$

Ex $x \in \mathbb{R}^n$ and $\varphi_1, \dots, \varphi_n \in \mathbb{R}^n$ ONB (orthonormal basis)

We can write

$x = \sum_{i=1}^{n} \alpha_i \varphi_i$ with $\alpha_i = \langle \varphi_i, x \rangle$.

Ex $\langle x, x \rangle = \sum_{i=1}^{n} \alpha_i^2$ [Parseval]

Ex: $\langle x, Ax \rangle = \sum_{i=1}^{n} \alpha_i^2 \lambda_i$

Ex A sym $\Rightarrow$ A has real eigenvalues

Def Rayleigh quotient $\frac{\langle x, Ax \rangle}{\langle x, x \rangle}$ (for $x \neq 0$)

Ex $Tr(A) = \sum_{i=1}^{n} \lambda_i$

Ex $Tr(A^k) = \sum_{i=1}^{n} \lambda_i^k$

Ex $Tr(A^2) = 2|E|$

Ex If G is simple with $\deg \geq 1$

$$\Downarrow$$

$$\lambda_n < 0$$

Ex Prove that $\frac{1}{n}\sum_{i=1}^{n} \lambda_i^2 = d$

Def $J_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$ all ones $n \times n$ matrix

Ex Compute eigenvalues of J_n/n

Def K_n complete graph on n vertices

Def $K_{a,b}$ complete bipartite graph
$G = (V = L \sqcup R, E)$ with $|L| = a, |R| = b$

Ex Compute the spectrum of K_n

Ex Compute the spectrum of $K_{n,n}$

Ex Compute " " of $K_{1,d}$

Ex " " " of $K_{a,b}$

Ex $\lambda_1 \geq \max_{i \geq 2} |\lambda_i|$

Ex G bipartite $\Longleftrightarrow$ $\mathrm{Spec}(G) = -\mathrm{Spec}(G)$.

Some Notions of Expansion

Def $\partial(S) = E(S, \bar{S})$ "Edge boundary"

Def $\Phi(S) = \dfrac{|\partial(S)|}{d|S|}$ "Conductance"

Def $\Phi(G) = \min\limits_{\substack{\emptyset \neq S \subseteq V \\ |S| \leq n/2}} \Phi(S)$ "Cheeger's constant"

Def: $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$

two-sided spectral expansion

Def $\lambda^{\text{one-sided}} = \lambda_2$ one-sided spectral expansion

Def $e(S,T) = |\{(s,t) \mid \{s,t\} \in E\}|$

Ex Prove that

$$\left| e(S,T) - \frac{d|S||T|}{n} \right| \le \lambda \sqrt{|S||T|}.$$

[Expander Mixing Lemma]

Ex Improve the error bound $\lambda\sqrt{|S||T|}$.

Def $\alpha(G)$ = independence number

Ex Prove that $\dfrac{\alpha(G)}{n} \le \dfrac{-\lambda_n}{d-\lambda_n}$

[Hoffman's bound]

Ex Prove that $d_{avg} \le \lambda_1 \le \Delta(G)$

Def $\chi(G)$ is the chromatic number

Ex Prove that $\chi(G) \le \lambda_1 + 1$

[Wilf's bound]

Mixing Bounds

Def $\vec{1}$ is the all ones vector

Def $R = \frac{1}{d} A$ is the random walk matrix

Ex $R \frac{\vec{1}}{n} = \frac{\vec{1}}{n}$

Ex Prove $\left\| R^r p - \frac{\vec{1}}{n} \right\|_1 \leq \left(\frac{\lambda}{d}\right)^r \sqrt{n}$ for any distribution p.

[Mixing bound in l_1]

Study the Perron-Frobenius theorem (useful for understanding more general Markov chains)

Eigenvalues as an optimization problem

Let $V_k = \text{span}\{e_1, \dots, e_k\}$
$W_k = \text{span}\{e_k, \dots, e_n\}$

Ex $\lambda_k = \min_{0 \neq x \in V_k} \frac{\langle x, Ax \rangle}{\langle x, x \rangle} = \max_{0 \neq x \in W_k} \frac{\langle x, Ax \rangle}{\langle x, x \rangle}$

Ex Prove the min-max variational theorem

$$\lambda_k = \max_{\substack{V \subseteq \mathbb{R}^n \\ \dim(V) = k}} \min_{0 \neq x \in V} \frac{\langle x, Ax \rangle}{\langle x, x \rangle} = \min_{\substack{V \subseteq \mathbb{R}^n \\ \dim(V) = n-k+1}} \max_{0 \neq x \in V} \frac{\langle x, Ax \rangle}{\langle x, x \rangle}$$

[Courant-Fischer-Weyl]

The Magic of Interlacing

Ex Eigenvalue Interlacing

Let $A \in \mathbb{R}^{n \times n}$ be real symmetric matrix and B be a $(n-1) \times (n-1)$ <u>principal</u> submatrix

$$eig(A) = \{\lambda_1 \geq \dots \geq \lambda_n\}$$

$$eig(B) = \{\tilde{\lambda}_1 \geq \dots \geq \tilde{\lambda}_{n-1}\}$$

Prove $\lambda_1 \geq \tilde{\lambda}_1 \geq \lambda_2 \geq \dots \geq \tilde{\lambda}_{n-1} \geq \lambda_n$

Hint: use min-max theorem for eigenvalues

Extend to $r \times r$ principal submatrix B with $1 \leq r < n$

Ex: $\lambda_j \geq \tilde{\lambda}_j \geq \lambda_{j+n-r}$ for $j \in \{1, \dots, r\}$

[Cauchy Interlacing Thm]

Refresher on PSDness

Def A real sym matrix M is *positive semi-definite* (PSD) if

$\forall x \in \mathbb{R}^n, \quad x^t M x \geq 0.$

Ex Prove: The following are equivalent

1) M is PSD
2) M has non-negative eigenvalues
3) $\exists$ a matrix W s.t. $M = W^t W$

Notation We write $M \succcurlyeq 0$ if M is PSD

We write $M_1 \succcurlyeq M_2$ if $M_1 - M_2 \succcurlyeq 0$.

This gives a partial order (Loewner order)

Laplacian Matrix

Def $L = dI - A$ [Laplacian Matrix]

Let $\mu_1 \le \mu_2 \le \dots \le \mu_n$ be the eigenvalues of L

Ex For d-regular G, we have

$$\underset{\substack{\parallel \\ 0}}{\mu_1 = d - \lambda_1}, \quad \dots, \quad \mu_n = d - \lambda_n.$$

Ex Prove that $\langle x, Lx \rangle = \sum_{i \sim j} (x_i - x_j)^2$

Ex Conclude that $L \succcurlyeq 0$ (PSD)

Ex Prove that $\langle 1_S, L 1_S \rangle = |E(S, \bar{S})|$

Ex G connected $\Longleftrightarrow$ $\mu_2 > 0$

Ex If G is connected, then $\mu_2 \geq \frac{1}{n \operatorname{diam}(G)}$

Ex $|\{k \mid \mu_k = 0\}|$ = # of connected components

Ex G bipartite iff $\mu_n = 2d$

Ex Prove that $\frac{\mu_2}{2d} \leq \Phi(G)$

$\boxed{CH^*}$ Prove that $\Phi(G) \leq O\left(\sqrt{\frac{\mu_2}{d}}\right)$

Hint: use eigenvector to μ_2 to find a cut ["rounding"]

[Cheeger's Inequality]

$$\frac{\mu_2}{2d} \leq \Phi(G) \leq \sqrt{\frac{2\mu_2}{d}}$$

Characteristic Polynomial

Def Characteristic polynomial $\det(\lambda \underline{I} - A) =: ch(\lambda)$

The roots of $ch(\lambda)$ are the eigenvalues of A

Cayley-Hamilton Theorem: $ch(A) = 0$

Ex G connected has diam = K $\Rightarrow$ A has at least K+1 distinct eigenvalues

Hint: [Cayley-Hamilton] minimal polynomial

Ex Let A, B be two real sym matrices with eig $(\lambda_1 \geq \cdots \geq \lambda_n)$
$(\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_n)$

Compute the eigenvalues of $A \otimes B$.

Def $s: E \to \{\pm 1\}$ is an edge signing

Def $(A_s)_{u,v} = \begin{cases} s(\{u,v\}) & \text{if } \{u,v\} \in E \\ 0 & \text{otherwise} \end{cases}$

Ex Prove $\lambda_1(A_s) \leq \Delta(G)$ for any signing s.

Limitations on Spectral Expansion

Ex G d-regular $\Rightarrow \lambda \geq \sqrt{d}\,(1 - o_n(1))$

Ex If G has diam $\geq 4 \Rightarrow \lambda_2 \geq \sqrt{d}$
[Hint: look for the star]

Ex $\lambda_2 < 0 \iff G = K_n$
[Hint: interlacing]

Ex Suppose G is connected. G has a unique positive eigenvalue iff G is a complete k-partite graph
[Hint: interlacing]

$\boxed{\text{Ch}^*}$ $\lambda_2 \geq 2\sqrt{d-1}\left(1 - O\left(\frac{1}{\text{diam}}\right)\right)$

[Alon-Boppana bound]

Cor) If $d = O(1)$

$$\Downarrow$$

$$\lambda_2 \geq 2\sqrt{d-1}\left(1 - O\left(\frac{1}{\log n}\right)\right)$$

Def G is <u>Ramanujan</u> if $\lambda \leq 2\sqrt{d-1}$

a.k.a. "Optimal" Spectral Expanders

OP [van Gogh Prize]

Construct infinite families of Ramanujan graphs for <u>every</u> $d \geq 3$

Vertex Expansion

Def $N(S) = \{u \mid \exists s \in S, \{u,s\} \in E\}$

Def Vertex (or Lossless) Expansion

$$\Phi^V(S) = \frac{|N(S)|}{d|S|}$$

$$\Phi^V_\varepsilon(G) = \min_{\substack{\emptyset \neq S \subseteq V \\ |S| \leq \varepsilon n}} \Phi^V(G).$$

OP [van Gogh prize]

Construct explicit family with

$$\Phi^V_\varepsilon(G) > \frac{1}{2} \quad \text{for } \varepsilon = \Omega(1)$$

(or two-sided bipartite lossless)

On the Complexity of Expansion

(Hypothesis) $\forall\, \eta \in (0,1)\ \exists\, \delta \in (0,1)$ s.t. it is NP-hard to distinguish given input graph $G=(V,E)$

(Yes) $\exists\, S \subseteq V$ with $|S| \le \delta n$ and $\Phi(S) \le \eta$.

(No) $\forall\, S \subseteq V$ with $|S| \le \delta n$, we have $\Phi(S) \gtrsim 1-\eta$

OP van Gogh prize

Prove or refute the above hypothesis

Def: Boolean n-hypercube is the graph $H_n = (V, E)$ where

$$V = \mathbb{Z}_2^n$$

$$E = \{\{u,v\} \mid |u-v| = 1\}$$

or equivalently

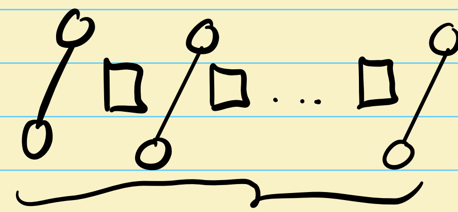
$$E = \{\{u, u+e_i\} \mid u \in \mathbb{Z}_2^n, i \in [n]\}$$

Ex Show that the adjacency matrix of H_n can be defined recursively as

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_n = \left(\begin{array}{c|c} A_{n-1} & I_{2^{n-1}} \\ \hline I_{2^{n-1}} & A_{n-1} \end{array}\right).$$

Def The cartesian product of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is defined as

$$G \square H = \Big(V = V(G) \times V(H),$$
$$E = \{\{(g_1, h_1), (g_2, h_2)\} \mid$$
$$(\{g_1, g_2\} \in E(G) \text{ and } h_1 = h_2) \text{ or }$$
$$(g_1 = g_2 \text{ and } \{h_1, h_2\} \in E(H))\}\Big)$$

Ex Prove that $H_n =$ 

n times

Ex Prove that "the" adjacency matrix of $G \square H$ can be written as

$$A_{G \square H} = A_G \otimes I_{|V(H)|} + I_{|V(G)|} \otimes A_H$$

Ex Prove that

$$\mathrm{Spec}(A_{G \square H}) = \{ \lambda + \tilde{\lambda} \mid \lambda \in \mathrm{Spec}(G), \tilde{\lambda} \in \mathrm{Spec}(H) \}$$

Ex Compute $\mathrm{Spec}(H_n)$

Ex Consider the recursive edge signing of H_n

$$B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_n = \left(\begin{array}{c|c} B_{n-1} & I_{2^{n-1}} \\ \hline I_{2^{n-1}} & -B_{n-1} \end{array}\right)$$

Prove that B_n has eigenvalues $\pm\sqrt{n}$ each with multiplicity 2^{n-1}.

[Hint: show $B_n B_n = n I_{2^n}$]

Def The induced subgraph of $G=(V,E)$ on $S \subseteq V$ is defined as

$$G[S] = (S, E' = \{\{u,v\} \in E \mid u,v \in S\})$$

Ex Prove that

$$(\forall S \subseteq V(H_n) \text{ with } |S| \geq 2^{n-1}+1)$$

$$(\Delta(H_n[S]) \geq \sqrt{n})$$

[Hint: Cauchy interlacing on B_n and $\lambda_1(G) \leq \Delta(G)$]

[Huang's theorem] implies the Sensitivity Conjecture via a known connection of Gotsman and Linial

Recall

OP van Gogh prize

Show that (families) of Ramanujan graphs exist for <u>every</u> degree $d \geq 3$.

OP van Gogh prize

Signing Conjecture [Bilu-Linial]

Every d-regular graph $G=(V,E)$ has an edge signing $s: E \to \{\pm 1\}$ such that the signed adjacency matrix satisfies $\mathrm{eig}(A_s) \subseteq [-2\sqrt{d-1}, 2\sqrt{d-1}]$.

(Positive answer would resolve the first OP on this page.)

Fourier Analysis

Def Let $S \subseteq [n]$. The character $\chi_S : \mathbb{Z}_2^n \to \{\pm 1\}$ is defined as

$$\chi_S(x) = \prod_{i \in S} (-1)^{x_i}.$$

Let $f, g : \mathbb{Z}_2^n \to \mathbb{R}$.

Def $$\langle f, g \rangle = \mathbb{E}_{x \in \mathbb{Z}_2^n} f(x) g(x)$$

(This is just a convenient normalization for Fourier analysis (different from before).)

Ex $$\langle \chi_S, \chi_T \rangle = \begin{cases} 1 & S = T \\ 0 & o/w \end{cases}$$

Ex $\{\chi_S\}_{S \subseteq [n]}$ forms an ONB for the space of functions $\{f : \mathbb{Z}_2^n \to \mathbb{R}\}$.

Ex $\exists!$ Fourier decomposition

$$f = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S$$

where $\hat{f}(S) := \langle f, \chi_S \rangle$.

Ex $\chi_S(x+y) = \chi_S(x) \chi_S(y)$

[homomorphism $\mathbb{Z}_2^n \to \{\pm 1\}$]

Ex $\langle f, f \rangle = \sum_{S \subseteq [n]} \hat{f}(S)^2$

[Parseval]

Ex $\chi_\emptyset = 1$

Ex $\hat{f}(\emptyset) = \mathbb{E}_{x \in \mathbb{Z}_2^n} f(x)$

Ex $\mathrm{Var}[f] = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} \hat{f}(S)^2$

Def The convolution of f and g is defined as

$$f * g(x) = \mathbb{E}_{y \in \mathbb{Z}_2^n} f(y) g(x-y).$$

Ex $\widehat{(f*g)}(S) = \hat{f}(S) \cdot \hat{g}(S)$

Def The degree of f is

$$\deg(f) = \max_{S : \hat{f}(S) \neq 0} |S| .$$

Def $\mathrm{dist}(f,g) = \Pr_{x \in \mathbb{Z}_2^n} [f(x) \neq g(x)]$

Ex If $f, g : \mathbb{Z}_2^n \to \{\pm 1\}$, then

$$\langle f, g \rangle = 1 - 2\,\mathrm{dist}(f,g) .$$

Ex Let A be the adjacency matrix of H_n. Prove that

$$A \chi_S = (n - 2|S|) \chi_S$$

[Characters as eigenvectors]

Def 1 $f : \mathbb{Z}_2^n \to \underline{\mathbb{Z}_2}$ is linear if

$$f(x) = \sum_{i=1}^{n} c_i x_i \quad \text{for some } c \in \mathbb{Z}_2^n.$$

Def 2 $f : \mathbb{Z}_2^n \to \underline{\mathbb{Z}_2}$ is linear if

$$f(x) + f(y) = f(x+y) \quad \forall x, y \in \mathbb{Z}_2^n$$

Ex Def 1 $\iff$ Def 2.

Property Testing Model

Query

$$\underset{\mathbb{Z}_2^n}{\overset{x}{\cap}} \longrightarrow \boxed{f : \mathbb{Z}_2^n \to \mathbb{Z}_2} \longrightarrow f(x)$$

Def A property is a subset P of functions from $\{f : \mathbb{Z}_2^n \to \mathbb{Z}_2\}$

E.g. $P_{Lin} = \{\text{linear functions}\}$

Def $\operatorname{dist}(f, P) = \min_{g \in P} \operatorname{dist}(f, g)$

Meta Question: Decide

(1) f has property P or

(2) f is ε-far from P.

Consider the following 3-query tester for linearity

(1) Sample $x, y \in \mathbb{Z}_2^n$ uniformly

(2) Accept iff $f(x) + f(y) = f(x+y)$.

Ex If f is linear, then Tester accepts with probability 1.

For convenience let's think f maps to $\{\pm 1\}$ instead of $\mathbb{Z}_2$.

Def $P_{rej}(f) = \Pr[\text{Tester rejects } f]$

Ex $\mathbb{E}_{x,y \in \mathbb{Z}_2^n} f(x) f(y) f(x+y) = 1 - 2 p_{rej}(f)$

Ex $p_{rej}(f) \geq dist(f, P_{Lin})$

[Hint: Fourier analysis and convolution]

[BLR linearity testing]

Ex $\chi_S \cdot \chi_T = \chi_{S \Delta T}$

Complexity Measures

Let $f: \mathbb{Z}_2^n \to \{\pm 1\}$, $x \in \mathbb{Z}_2^n$

Def sensitivity of f at x

$$s(f,x) = |\{ i \in [n] \mid f(x) \neq f(x+e_i)\}|$$

Def sensitivity of f

$$s(f) = \max_{x \in \mathbb{Z}_2^n} s(f,x)$$

Notation Let $B \subseteq [n]$. $x^B = x + \sum_{i \in B} e_i$

Def block sensitivity of f at x

$$bs(f,x) = \max\{ K \mid \exists \text{ disjoint } B_1, \dots, B_K \subseteq [n] \text{ s.t. } f(x) \neq f(x^{B_i}) \ \forall i \in [K]\}$$

Def block sensitivity of f

$$bs(f) = \max_{x \in \mathbb{Z}_2^n} bs(f,x)$$

Ex $bs(f) \geq s(f)$

Ex* find f satisfying $bs(f) \geq \Omega(s(f)^2)$

Recall $\deg(f) = \max_{S : \hat{f}(S) \neq 0} |S|$

We say a polynomial $p(x_1, \dots, x_n)$ $\frac{1}{3}$-approximates $f : \{\pm 1\}^n \to \{0,1\}$ if

$$|p(x) - f(x)| \leq \frac{1}{3} \quad \forall x \in \{\pm 1\}^n$$

Def Approximate degree

$$\widetilde{\deg}(f) = \min\{\text{degree}(p) \mid p \ \tfrac{1}{3}\text{-approximates } f\}$$

Classical query model (viewed as a string)

$$j \longrightarrow \boxed{x \in \{\pm 1\}^n} \longrightarrow x_j$$

[Think of n as very large]

Quantum query model (")

$$|j\rangle \longrightarrow \boxed{x \in \{\pm 1\}^n} \longrightarrow x_j\,|j\rangle$$

O_x unitary

$$O_x|j\rangle = x_j|j\rangle$$

Decision Tree: Simplified computational model

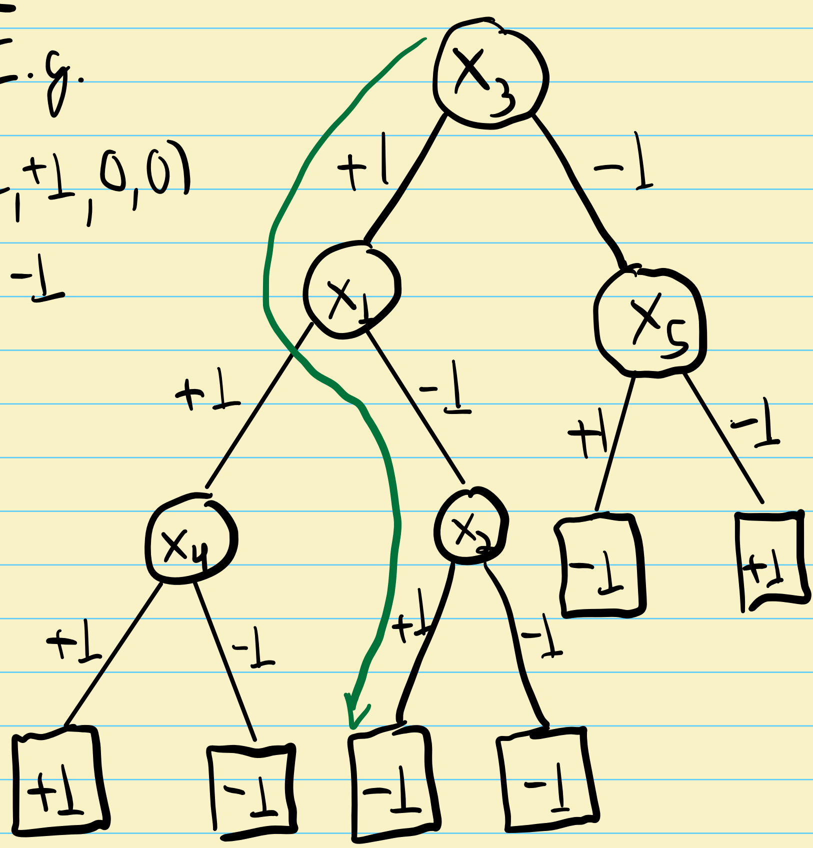
Input: $x = (x_1, \dots, x_n)$

Internal nodes label query variables

E.g.

$x = (-1, +1, +1, 0, 0)$

output: -1



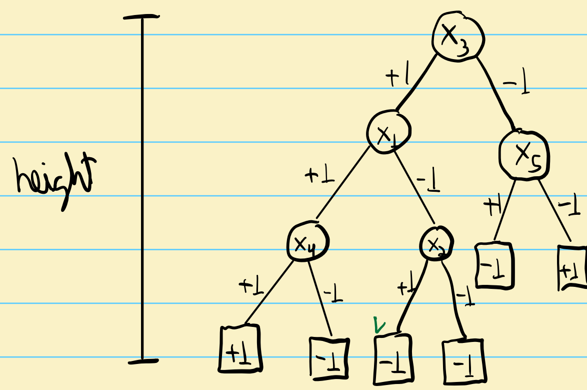
Start at the root, and query input variables until leaf is reached

Leaves are labelled with output value of a computational path

Output: valued of leaf reached by evaluating the decision tree

Def height(Tree) = length of a longest path from root to a leaf

E.g. for the previous example the height is 3



Def Decision tree complexity of f (or classical query complexity)

$$D(f) = \min\{\text{height}(T) \mid T \text{ is a tree computing } f\}$$

Ex $D(f) \geq \deg(f)$

Quantum Query Model

Given $f: \mathbb{Z}_2^n \to \{0,1\}$, a t-query protocol specifies $t+1$ unitaries $U_0, \dots, U_t$,

$$\Big(U_t O_x \cdots O_x U_1 O_x U_0 |0\rangle \Big),$$

starting state

t queries to the oracle

Π_{acc} : projector onto accepting subspace,

$$p_{acc}(x) := \| \Pi_{acc} U_t O_x \cdots O_x U_1 O_x U_0 |0\rangle \|^2$$

such that

$$|p_{acc}(x) - f(x)| \leq 1/3 \quad \forall\, x \in \mathbb{Z}_2^n$$

Ex $p_{acc}(x)$ is a polynomial of degree $\leq 2t$ in x.

Def Quantum query complexity of f

$Q(f) = \min\{t \mid \exists$ quantum t-query protocol for $f\}$

Ex $Q(f) \geq \frac{\widetilde{\deg}(f)}{2}$

Before 2018

$s(f)$

not known to be poly-related to

$bs(f)$

$\deg(f)$

$\widetilde{\deg}(f)$ $\quad \ldots \; R(f) \ldots$

$D(f)$

$Q(f)$

all polynomially related

NS'90 [Sensitivity Conjecture]

$$bs(f) \le \mathrm{poly}(s(f))$$

Proved by Huang in 2018

Ex $D(f) \geq Q(f)$

No superpolynomial quantum speed-up for total functions in the query model of computation

$\boxed{\text{CH}}$ If f is a total function, then

$$Q_\epsilon(f) \geq D(f)^{\Theta(1)}.$$

OP Van Gogh prize

Find other efficient quantum algorithms for "useful" tasks with no known efficient classical algorithm (give "evidence" that none exist)

Group Theory Refresher

[notation overload]

A group $(G, \circ)$ is a set G with a binary operation '$\circ$' satisfying

1) $g_1 \circ g_2 \in G \quad \forall g_1, g_2 \in G$

2) $\exists\, 1 \in G$ s.t. $1 \circ g = g \circ 1 = g \quad \forall g \in G$

(existence of identity)

3) $\forall g \in G\ \exists\, g^{-1} \in G$ s.t. $g \circ g^{-1} = g^{-1} \circ g = 1$

(existence of inverse)

4) $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 \quad \forall g_1, g_2, g_3 \in G$

(associativity)

(We may use $g_1 g_2$ to denote $g_1 \circ g_2$)

We say that $(G, \circ)$ is <u>Abelian</u> if

$$g_1 \cdot g_2 = g_2 \cdot g_1 \quad \forall\, g_1, g_2 \in G.$$

(commutativity)

In this case, we may use '+' instead of '$\circ$'.

We may simply say G is group.

Cayley Graphs

Let G be a group and $S \subseteq G$.

$Cay(G,S)$ is the graph with vertex set $V=G$ and (directed) edge set

$E=\{(g,sg) \mid g \in G, s \in S\}$

Ex $H_n = Cay(\mathbb{Z}_2^n, \{e_1, \ldots, e_n\})$

Ex $Cay(G,G)$ is the complete graph with self-loops

Ex $Cay(G, G \setminus \{1\})$ is $K_{|G|}$

Ex $Cay(G, \{1\})$ is the graph with only self-loops

Ex If $S = S^{-1}$, then $Cay(G, S)$ is undirected.

Ex The cycle graph C_n on n vertices is $Cay(\mathbb{Z}_n, \{\pm 1\})$.

Ex Generalize the definition of a character from $\mathbb{Z}_2$ to $\mathbb{Z}_3$ defining maps $\chi : \mathbb{Z}_3 \to \mathbb{C}$ that are homomorphisms to the complex unit circle

Ex Same question from before but now from $\mathbb{Z}_3$ to $\mathbb{Z}_n$

Ex If χ and ψ are characters, then so is $\chi \cdot \psi$.

Ex $\chi = 1$ is a (trivial) character

Ex If $\chi \neq 1$, then $\sum_{g \in G} \chi(g) = 0$.

Ex $\chi(g^{-1}) = \bar{\chi}(g)$

Ex $\frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)}\, \psi(g) = \begin{cases} 1 & \text{if } \chi = \psi \\ 0 & \text{o/w} \end{cases}$

Ex* Generalize the definition of character to finite Abelian groups

Let A be the adjacency matrix of $Cay(G, S)$ for some Abelian group G. Let $f : G \to \mathbb{C}$.

Ex $(Af)(x) = \sum_{s \in S} f(sx)$

Ex If χ is a character, then χ is an eigenvector

$$A\chi = \lambda \chi$$

with eigenvalue

$$\lambda = \sum_{s \in S} \chi(s) .$$

Ex Compute the eigenvalues of C_n

Ex Compute $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$ of C_n

Is it a good expander or not? (for large n)

Ex Compute the Cheeger constant of C_n.

Ex Define a "right" Cayley graph with multiplication by a generator on the right

Ex Given two sets of generators $A, B \subseteq G$ define a "left-right" Cayley graph

$$\mathrm{Cay}(A, G, B)$$

Ex Find a "squares" in $\mathrm{Cay}(A, G, B)$

Hint:

$$\begin{tikzcd}
gb \arrow[r, dash] \arrow[d, dash] & agb \arrow[d, dash] \\
g \arrow[r, dash] & ag
\end{tikzcd}$$

$g \in G$

$a \in A$

$b \in B$

Pseudorandom Distributions

$\mathcal{F} \subseteq \{ f : \mathbb{F}_2^n \to \mathbb{F}_2 \}$ be a collection of functions to be "fooled"

Def We say that a distribution $\mathcal{D}$ on $\mathbb{F}_2^n$ ε-fools $\mathcal{F}$ if $\forall f \in \mathcal{F}$

$$\left| \Pr_{x \in \mathbb{F}_2^n} [f(x) = 1] - \Pr_{x \sim \mathcal{D}} [f(x) = 1] \right| \leq \varepsilon/2 .$$

Def We say that a distribution $\mathcal{D}$ on $\mathbb{F}_2^n$ is ε-biased if it ε-fools $\mathcal{F} = \{ \chi_S \mid S \subseteq [n] \}$ (all characters),

Equivalently

$$\left| \mathbb{E}_{x \sim \mathcal{D}} \chi_S(x) \right| \leq \varepsilon \quad \forall \; \begin{matrix} S \neq \emptyset \\ S \subseteq [n] \end{matrix} .$$

Suppose $\mathcal{D}$ is an ε-biased distribution that is uniform on some multiset of $\mathbb{F}_2^n$

Ex Use $\mathcal{D}$ to define

$$\mathrm{Cay}(\mathbb{Z}_2^n, \underbrace{?}_{\text{generating set}})$$

s.t. $\lambda(\mathrm{Cay}(\mathbb{Z}_2^n, \underline{?})) \leq \varepsilon$.

(normalized) two-sided spectral expander

Ex Provide a converse transformation

Equivalent pseudorandom objects

ε-biased dist. $\Longleftrightarrow$ ε-expander

Coding Theory

$\Sigma = \{1, \dots, q\}$ alphabet (q-ary)

Def Any $\mathcal{C} \subseteq \Sigma^n$ is a code

n is the blocklength

Def (Normalized Hamming Distance) $x, y \in \Sigma^n$

$$\Delta(x,y) = \frac{1}{n} \sum_{i=1}^{n} 1_{[x_i \neq y_i]} \in [0,1]$$

Fundamental Parameters of $\mathcal{C}$

Def (Minimum Distance of $\mathcal{C}$)

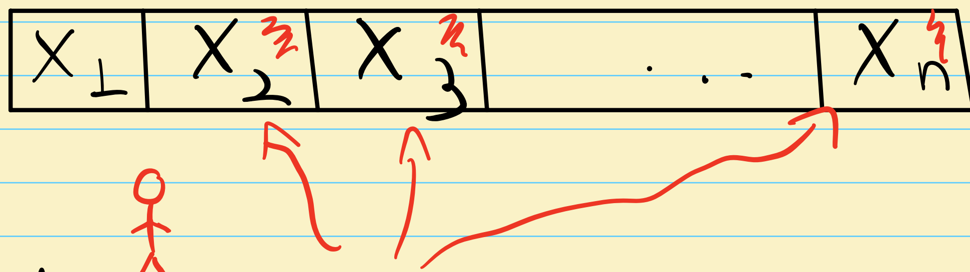
$$\Delta(\mathcal{C}) = \min_{\substack{x,y \in \mathcal{C} \\ x \neq y}} \Delta(x,y) \in [0,1]$$

Def (Rate)

$$r(\mathcal{C}) = \frac{\log_q(|\mathcal{C}|)}{n} \in [0,1]$$

Error Model (Hamming)

$x \in \mathcal{C} \subseteq \Sigma^n$



Adversary can change symbols of x leading to some corrupted $\tilde{x} \in \Sigma^n$.

Def (Error) $x_i \neq \tilde{x}_i$

Ex If $\Delta(\mathcal{C}) = d/n$, then $\mathcal{C}$ can correct $\leq \left\lfloor \frac{d-1}{2} \right\rfloor$ adversarial errors

Coding Theory "Wishlist"

Want $\mathcal{C} \subseteq \Sigma^n$ satisfying

- Both $\Delta(\mathcal{C})$ and $r(\mathcal{C})$ as large as possible (best rate-vs-distance trade-off)
- Efficient encoding
- Efficient (list) decoding
- Explicit construction
- Over small alphabets (ideally binary)
- Local properties (local testability, decodability, etc)

Achieving subsets of these wishes is wide open in many cases!

Def (Linear Code)

We say $\mathcal{C}$ is a linear code if $\Sigma = \mathbb{F}_q$ and $\mathcal{C} \subseteq \mathbb{F}_q^n$ is a <u>linear subspace</u>

Ex If $\mathcal{C}$ is linear and $\dim(\mathcal{C}) = k$, then $r(\mathcal{C}) = \frac{k}{n}$

Def (Normalized Hamming Weight) $x \in \Sigma^n$

$$|x| = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{[x_i \neq 0]}$$

Ex If $\mathcal{C}$ is linear, then

$$\Delta(\mathcal{C}) = \min_{\substack{x \in \mathcal{C} \\ x \neq 0}} |x|$$

Def $G \in \mathbb{F}_q^{n \times k}$ is a generating matrix of $\mathcal{C}$ if

$$\mathcal{C} = \operatorname{im}(G)$$

Def $H \in \mathbb{F}_q^{(n-k) \times n}$ is a parity check matrix of $\mathcal{C}$ if

$$\mathcal{C} = \operatorname{Ker}(H)$$

Ex If $\mathcal{C}$ is linear, $\mathcal{C}$ admits both a generating matrix and a parity check matrix.

Notation

$$\mathbb{F}_2^n = \{\alpha_1, \alpha_2, \ldots, \alpha_{2^n}\}$$

Def (Hadamard Code)

$$\mathcal{H}_n = \{(f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_{2^n})) \mid f : \mathbb{F}_2^n \to \mathbb{F}_2 \text{ is linear}\}$$

$N = 2^n$

Ex $\mathcal{H}_n \subseteq \mathbb{F}_2^N$ is a linear code

Ex $\Delta(\mathcal{H}_n) = \frac{1}{2}$

Ex $r(\mathcal{H}_n) = \frac{n}{N}$

Ex Write a generating matrix for $\mathcal{H}_n$

Ex Write a parity check matrix for $\mathcal{H}_n$

[Hint: BLR]

Ex Show that $|\mathcal{H}_n|$ is optimal among binary linear codes with $\Delta(\mathcal{C}) = \frac{1}{2}$.

Def (Dual Code) $\mathcal{C} \subseteq \mathbb{F}_q^n$ linear

$$\mathcal{C}^{\perp} = \{ y \in \mathbb{F}_q^n \mid \langle y, x \rangle = 0 \ \forall x \in \mathcal{C} \}$$

Ex $\mathcal{C}^{\perp}$ is a linear code

Ex $\dim(\mathcal{C}) + \dim(\mathcal{C}^{\perp}) = n$

Ex If H is the parity check of $\mathcal{C}$, then

$$\mathcal{C}^{\perp} = \text{row span}(H)$$

Ex H^{T} is the generating matrix of $\mathcal{C}^{\perp}$

Def (Polynomials of degree $\leq t$)

$$\mathbb{F}_q^{\leq t}[X] = \{ c_0 + c_1 X + \dots + c_t X^t \mid c_0, \dots, c_t \in \mathbb{F}_q \}$$

Let $k \leq n \leq q$

$\Gamma = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq \mathbb{F}_q$
(distinct)

Def (Reed - Solomon Code)

$$RS_\Gamma(k, n) = \{ (p(\alpha_1), \dots, p(\alpha_n)) \mid p \in \mathbb{F}_q^{\leq k-1}[X] \}$$

Extremely Important Fact

"Low degree polynomials have few roots"

Ex If $p \in \mathbb{F}_q^{\leq t}[X]$, then

$$|\{\alpha \in \mathbb{F}_q \mid p(\alpha) = 0\}| \leq t.$$

Ex $RS_\Gamma(k, n) \subseteq \mathbb{F}_q^n$ is a linear code

Ex $\Delta(RS_\Gamma(k,n)) \geq \frac{n-k+1}{n}$

Ex $r(RS_\Gamma(k,n)) = k/n$

Ex Write a generating matrix for $RS_r(k,n)$

[Hint: Vandermonde]

Ex Prove a rank lower bound on $l \times l$ submatrices (of the previous matrix) for $l \leq k$.

Ex Given $p \in \mathbb{F}_q^{k-1}[X]$, how many evaluation points are needed to interpolate it?

A General Code Upper Bound

Ex Any $\mathcal{C}$ with

$\Delta(\mathcal{C}) = d/n$ and

$r(\mathcal{C}) = k/n$ must

satisfy $k + d \leq n + 1$

(Singleton Bound)

Ex $RS_F(k, n)$ meets this bound

(optimality of RS codes)

Finite Fields: Properties

Ex $\alpha^q = \alpha \quad \forall \alpha \in \mathbb{F}_q$

$\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$

Ex $\exists \gamma \in \mathbb{F}_q^*$ s.t.

$$\mathbb{F}_q^* = \{\gamma^i \mid i \in \mathbb{N}\}$$

(Cyclic Property)

Ex $\sum_{\alpha \in \mathbb{F}_q} \alpha^i = 0$ for $1 \leq i < q-1$

Ex Compute $RS_{\Gamma}(k,n)^{\perp}$

Ex Write the parity check of $RS_{\Gamma}(k,n)$

Def (Good Codes)

We say that a family of codes (with $n \to \infty$) is <u>good</u> if

$$\Delta(\mathcal{C}) \geq \delta_0 = \Omega(1)$$

$$r(\mathcal{C}) \geq r_0 = \Omega(1)$$

for every $\mathcal{C}$ in the family

Def (ε-balanced code)

We say a linear binary code $\mathcal{C}$ is ε-balanced if $\forall\, 0 \neq x \in \mathcal{C}$

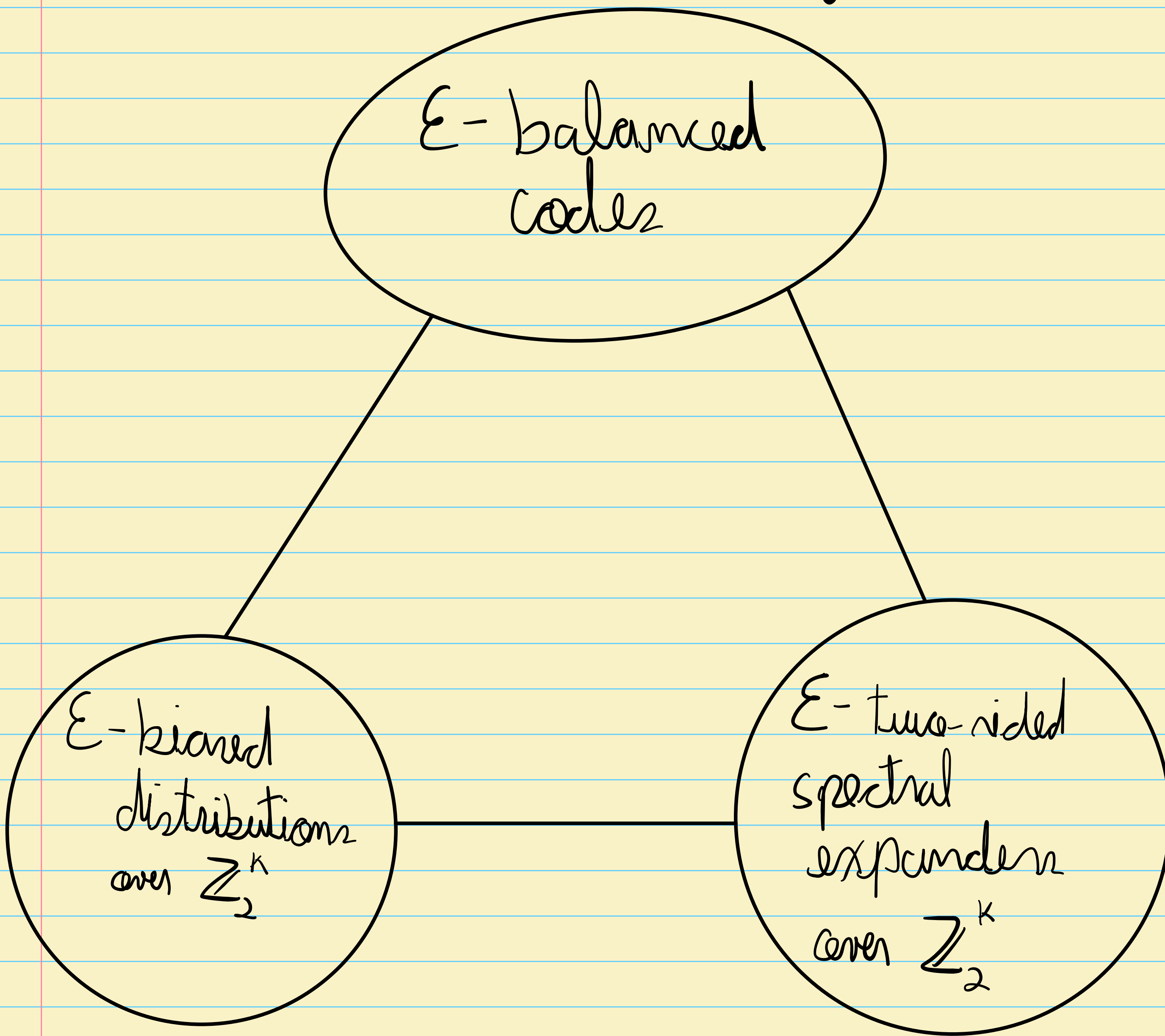
$$|x| \in \left[\frac{(1-\varepsilon)}{2}, \frac{(1+\varepsilon)}{2} \right].$$

Ex Starting from an ε-biased distribution $\mathcal{D}$ on $\mathbb{F}_2^k$ define an ε-balanced code $\mathcal{C} \subseteq \mathbb{F}_2^n$.

[Hint: $\dim(\mathcal{C}) = k$]

Equivalence Triad of Pseudorandom Objects

Ex Formalize all the equivalences:



Entropy and Volume

Def (Hamming Ball) $x \in \Sigma^n$, $\delta \in [0,1]$

$$B_q(x, \delta \cdot n) = \{ y \in \Sigma^n \mid \Delta(x,y) \le \delta \}$$

Ex $$|B_q(0, \delta \cdot n)| = \sum_{i=0}^{\lfloor \delta \cdot n \rfloor} \binom{n}{i} (q-1)^i$$

Ex Fix $\delta \in (0, \frac{1}{2}]$, find

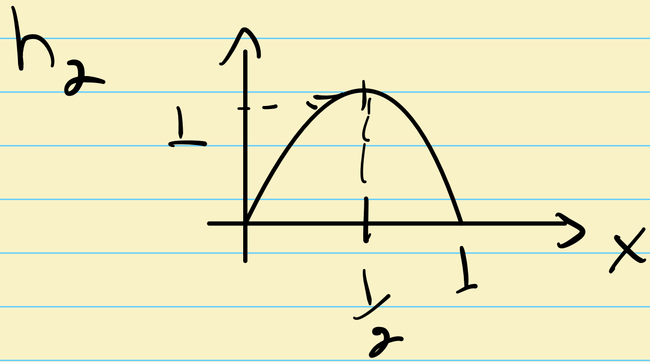
a function $f : [0,1] \to [0,1]$ s.t.

$$\lim_{n \to \infty} \frac{\log |B_2(0, \delta \cdot n)|}{f(\delta)\, n} = 1$$

[Hint: look ahead if very stuck]

Def (Binary Entropy) $h_2 : [0,1] \to [0,1]$

$$h_2(x) = x \log_2\left(\frac{1}{x}\right) + (1-x)\log_2 \frac{1}{(1-x)}$$



Ex $\delta \in (0, 1/2)$

$$2^{(h_2(\delta) - o_n(1))n} \le |B_2(0, \delta \cdot n)| \le 2^{h_2(\delta) n}$$

Ex Define the q-ary entropy by analogy with the binary case

Another Code Upper Bound

Ex If $\mathcal{C} \subseteq \Sigma^n$ with

$\Delta(\mathcal{C}) \geq d/n$, then

$$|\mathcal{C}| \leq \frac{q^n}{|B(0, \lfloor \frac{d-1}{2} \rfloor)|}.$$

(Hamming Bound)

$d = \delta \cdot n$ for $\delta \in [0, 1 - \frac{1}{q})$

Ex $r(\mathcal{C}) \leq 1 - h_q(\delta/2) + o_n(1)$

(Asymptotic Hamming Bound)

A Code Lower Bound

Ex Show that $\exists\, \mathcal{C} \subseteq \Sigma^n$ with

$$\Delta(\mathcal{C}) \geq d/n \quad \text{and}$$

$$r(\mathcal{C}) \geq 1 - \frac{\log_q(|B_q(0, d-1)|)}{n}$$

(Gilbert-Varshamov Bound)

or

GV Bound

Give two different proofs:

(1) Greedy Construction

(2) Random Construction

$d = \delta \cdot n \qquad \delta \in [0, 1 - 1/q)$

Ex $\exists\, \mathcal{C}$ with $\Delta(\mathcal{C}) \geq \delta$ and

$$r(\mathcal{C}) \geq 1 - h_q(\delta) - o_n(1)$$

(Asymptotic GV Bound)

Ex Show that a random <u>linear</u> code achieves the GV bound whp

(Hint: sample a generating matrix)

Ex If $\delta = \frac{(1-\varepsilon)}{2}$, then the code from the previous problem is ε-balanced w.h.p

OP van Gogh Prize

Construct explicit binary codes achieving the GV bound ($\delta \in [0, \frac{1}{2})$, $n \to \infty$)

OP van Gogh Prize

Is the GV bound asymptotically (up to $o_n(1)$ additive term) the best possible rate-vs-dist. trade-off for binary codes?

(Again for fixed $\delta \in [0, 1/2)$ and $n \to \infty$)

Tensor Codes

Let $C_A \subseteq \mathbb{F}_q^{n_A}$ and $C_B \subseteq \mathbb{F}_q^{n_B}$ be linear codes

Def (Tensor Code)

$$C_A \otimes C_B = \{ M \in \mathbb{F}_q^{n_A \times n_B} \mid \text{every col of } M \text{ belongs to } C_A \text{ and every row of } M \text{ belongs to } C_B \}$$

Ex $C_A \otimes C_B$ is linear

Ex Compute $\Delta(C_A \otimes C_B)$ in terms of $\Delta(C_A)$ and $\Delta(C_B)$

Ex Compute $r(C_A \otimes C_B)$ // $r(C_A)$ and $r(C_B)$

Ex $(C_A \otimes C_B)^{\perp} = \,?$

Ex $(C_A^{\perp} \otimes C_B^{\perp})^{\perp} = \,?$

Def (LDPC)

We say that $\mathcal{C}$ is a low-density parity check code if it admits a parity check matrix H with at most $b_0 = O(1)$ non-zero entries per row and per column.

(of course, this is for a family of codes)

Let $G = (V, E)$ be a n_0-regular <u>bipartite</u> graph with (normalized) one-sided spectral expansion $\lambda \in [0, 1)$.

(Think of n_0 as $O(1)$)

Let $\mathcal{C}_0 \subseteq \mathbb{F}_2^{n_0}$ be linear with

$$\Delta(\mathcal{C}_0) \geq \delta_0 \in (0, 1)$$

$$r(\mathcal{C}_0) \geq r_0 \in (0, 1)$$

Def For $v \in V(G)$, $\partial_G(v) := \{\{u,v\} \in E(G)\}$

Def (Tanner Code) (Tailored to this case)

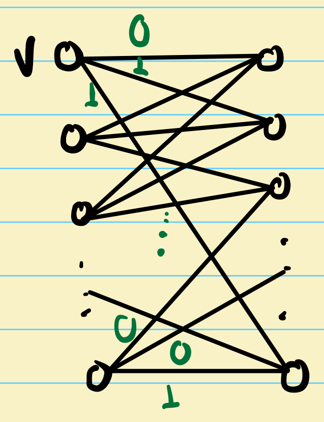
$$\text{Tanner}(G, \mathcal{C}_0) = \{x \in \mathbb{F}_2^E \mid x|_{\partial_G(v)} \in \mathcal{C}_0, \ \forall v \in V(G)\}$$

Obs: $x|_{\partial_G(v)}$ is viewed as a string in $\mathbb{F}_2^{d_0}$.
(Any ordering of the edges $\partial_G(v)$ will work for us)

Let $H_0 \in \mathbb{F}_2^{(1-r_0)d_0 \times d_0}$ be a parity check matrix for $\mathcal{C}_0$, i.e., $\mathcal{C}_0 = \text{Ker}(H_0)$

In a picture, $x \in \mathbb{F}_2^E$

$$x|_{\partial_G(v)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$



bits on the edges of G

$$H_0 \, x|_{\partial_G(v)} = 0$$
$$\Updownarrow$$
$$x|_{\partial_G(v)} \in \mathcal{C}_0$$

Local-to-global

Ex Write a parity check matrix for
Tanner$(G, \mathcal{C}_0)$

Ex Prove that if $d_0 = O(1)$, then
Tanner$(G, \mathcal{C}_0)$ is a LDPC code

Ex Compute a lower bound on

$$r(\text{Tanner}(G, \mathcal{C}_0))$$

[Hint: count constraints]

Ex Prove a lower bound on

$$\Delta(\text{Tanner}(G, \mathcal{C}_0)) \geq \delta_0(\delta_0 - \lambda)$$

[Hint: expander mixing lemma]

Ex Use explicit families of d_0-regular expander graphs to construct an explicit family of <u>good</u> binary <u>LDPC</u> codes

---

Ex Generalize the construction for codes over $\mathbb{F}_q$

Replacement Product

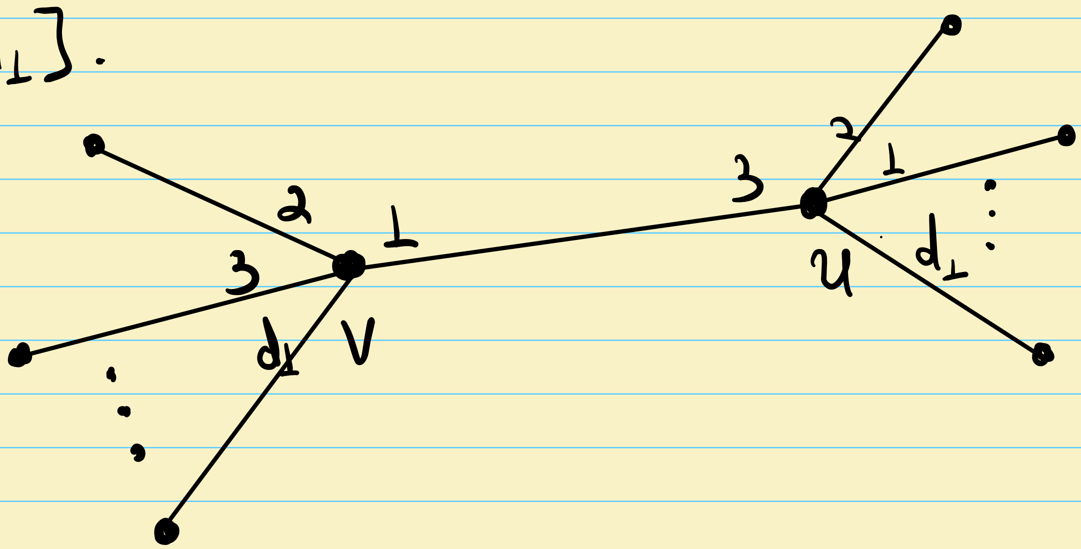
Let $G = ([n], E)$ be a d_1-regular graph

same

Let $H = ([d_1], E')$ be a d_2-regular graph

For each $v \in V(G)$, we label its incident edges with distinct numbers in $[d_1]$.

Eg



Using these labels we define a permutation on $V(G) \times [d_1]$

$P \in \mathbb{R}^{(V(G)\times[d_1])\times(V(G)\times[d_1])}$ s.t.

$$P_{(v,i),(u,j)} = \mathbb{1}\left[\begin{array}{l}\text{ith edge of } v \text{ is} \\ \text{incident to jth edge of } u\end{array}\right]$$

Ex $P^2 = I$

Obs Note that P is the adjacency matrix of a matching E matching

---

The <u>replacement product</u> of G and H is

$$G \circledr H = (V(G) \times V(H), E'') \text{ where}$$

$$E'' = E_{cloud} \sqcup E_{matching},$$

$$E_{cloud} = \bigsqcup_{v \in V(G)} E_v, \text{ and}$$

$$E_v = \{\{(v,i),(v,j)\} \mid i \sim_H j\}.$$

i and j are connected in H

Ex Draw $K_5 \circledr C_4$

Terminology: vertices $\{(v, i) \mid i \in [d_1]\}$ are said to be from the same "cloud"

Ex $G \circledR C_{d_1}$ is a 3-regular graph

Obs: $\circledR$ can lead to a great degree reduction!

Ex $G \circledR H$ is a (d_2+1)-regular graph

Zig-Zag Product

Let G be a (n, d_1, λ_1)-graph
$G = ([n], E)$

n: # of vertices
d_1: degree
λ_1: (normalized) two-sided spectral expansion

Let H be a (d_1, d_2, λ_2)-graph
$H = ([d_1], E')$

The <u>Zig-Zag product</u> of G and H

is defined by first considering

$G \textcircled{r} H$ with some permutation P s.t.

$$G \textcircled{z} H = (V(G) \times V(H), E'')$$

where ,

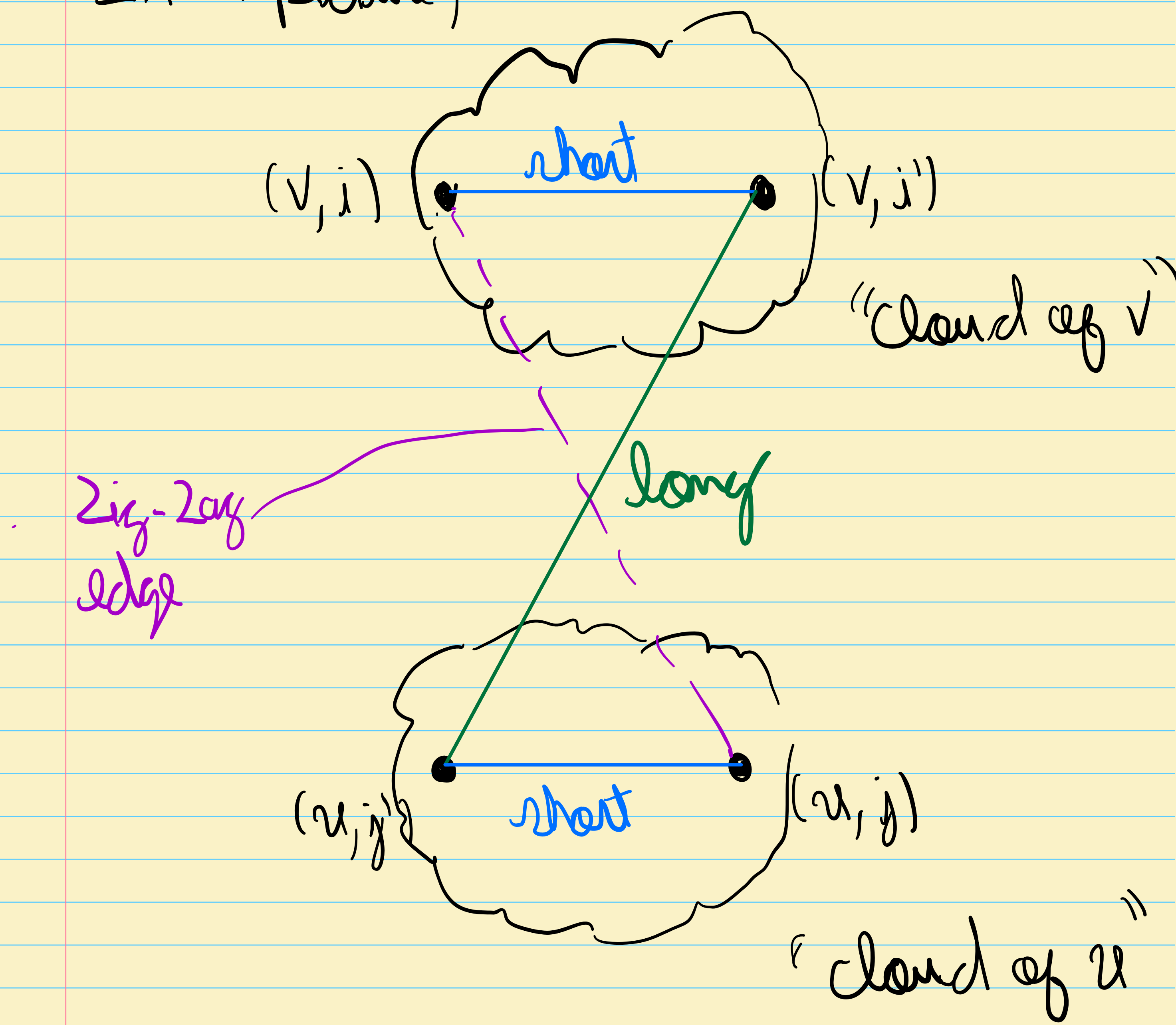
$$E'' = \{\{(v,i),(u,j)\} \mid \text{if } \exists\, i', j' \in V(H)$$

"short" $\quad (v,i) \underset{G \textcircled{r} H}{\sim} (v,i'),$

"long" $\quad (v,i') \underset{G \textcircled{r} H}{\sim} (u,j'),$ and

"short" $\quad (u,j') \underset{G \textcircled{r} H}{\sim} (u,j) \}$

In a picture,



Ex $G \textcircled{z} H$ is a d_2^2-regular graph on nd_1 vertices

---

Let A_G and A_H be the normalized adjacency matrices of G and H

Ex $A_{G \textcircled{z} H} = (I_{V(G)} \otimes A_H) P (I_{V(G)} \otimes A_H)$

Let $V^{\parallel} = \{ \vec{w} \otimes \vec{1} \in \mathbb{R}^{V(G) \times V(H)} \mid \vec{w} \in \mathbb{R}^{V(G)} \}$

Let $V^{\perp} \subseteq \mathbb{R}^{V(G) \times V(H)}$ be the orthogonal complement of $V^{\parallel}$

Ex $\left\langle \vec{w} \otimes \frac{\vec{1}}{\sqrt{d_1}}, P\left(\vec{w} \otimes \frac{\vec{1}}{\sqrt{d_1}}\right) \right\rangle$

$$\|$$

$$\langle \vec{w}, A_G \vec{w} \rangle$$

"Simulation Lemma" (uses the "cloud entropy")

Ex $\|P\|_{op} = 1$

Ex $\|I \otimes A_H\|_{op} = 1$

Ex Let $\vec{v} \in V^{\perp}$. Prove that

$$\|(I \otimes A_H)\vec{v}\|_2 \leq \lambda_2 \|\vec{v}\|_2$$

Ex $(I \otimes A_H)(\vec{w} \otimes \vec{1}) = \vec{w} \otimes \vec{1}$

Ex Prove

$$\max_{\substack{x \in \mathbb{R}^{V(G) \times V(H)} \\ x \perp \vec{1} \\ \|x\|_2 = 1}} \|A_{G \otimes H} x\|_2 \leq f(\lambda_1, \lambda_2)$$

for some f s.t. $f(\lambda_1, \lambda_2) \xrightarrow[\lambda_1, \lambda_2 \to 0]{} 0$

[Hint: use the previous problems]

Ex Conclude that $G \circled{z} H$ is a $(nd_1, d_2^2, f(\lambda_1, \lambda_2))$ graph

---

Now, we will consider a different analysis of the Zig-Zag

Ex $(I_{V(G)} \otimes J/d_1) P (I_{V(G)} \otimes J/d_1)$

$$\|$$

$$A_G \otimes J/d_1$$

Ex $A_H = (1-\lambda)\, J/d_\perp + \lambda E$

where $\|E\|_{op} \leq 1$

Ex Analyze again the second largest eigenvalue in absolute value

of

$A_{G \textcircled{z} H} = (I_{V(G)} \otimes A_H)\, P\, (I_{V(G)} \otimes A_H)$

using the previous problems

Ex Obtain a bound $g(\lambda_1, \lambda_2)$ s.t.

$$g(\lambda_1, \lambda_2) < 1 \text{ if }$$
$$\lambda_1 < 1 \text{ and } \lambda_2 < 1.$$

Ex Conclude that $G \text{ Ⓩ } H$ is a $(nd_1, d_2^2, g(\lambda_1, \lambda_2))$ graph

To be continued...