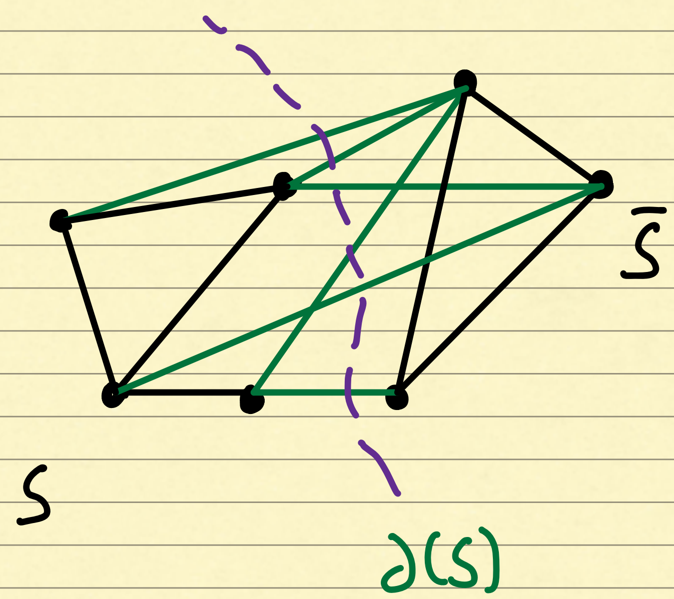
An Invitation to Expander Graphs

(Fernando Granha Jeronimo)
UIUC



Plan

1) What is an expander?

2) Warm up: Expander Mixing Lemma

3) Some Applications

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1) What is an expander?

2) Warm up: Expander Mixing Lemma

3) Some Applications

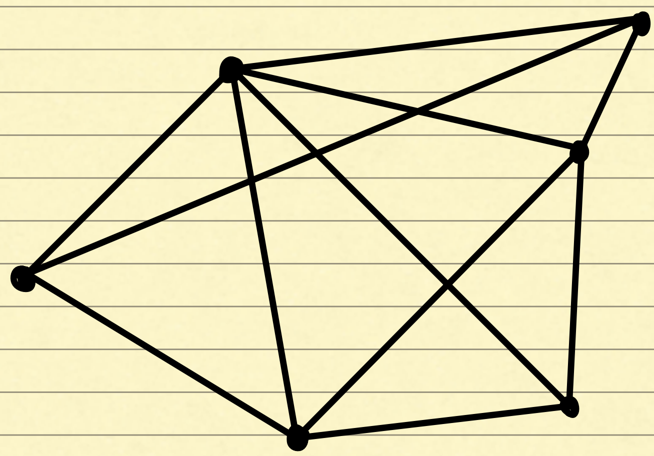
4) Zig-Zag Product

5) Near-optimal Expanders

What is an expander?

Informal Def: A well connected yet sparse graph

Graph $\quad G = (V, E)$



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"perspectives"

- Combinatorial
- Algebraic
- Random Walk

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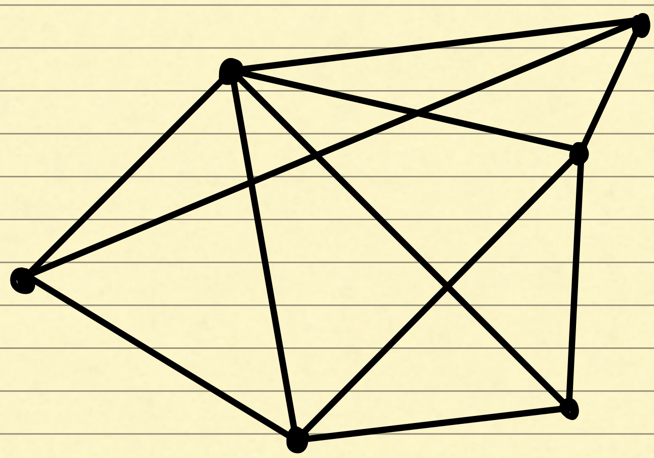

What is an expander?

Informal Def: A well connected yet sparse graph

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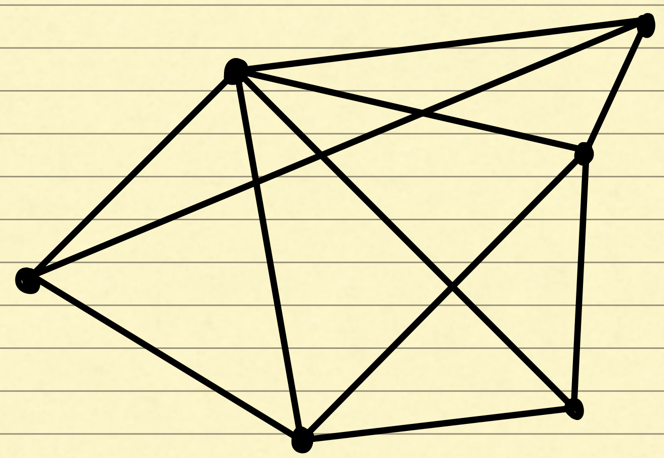
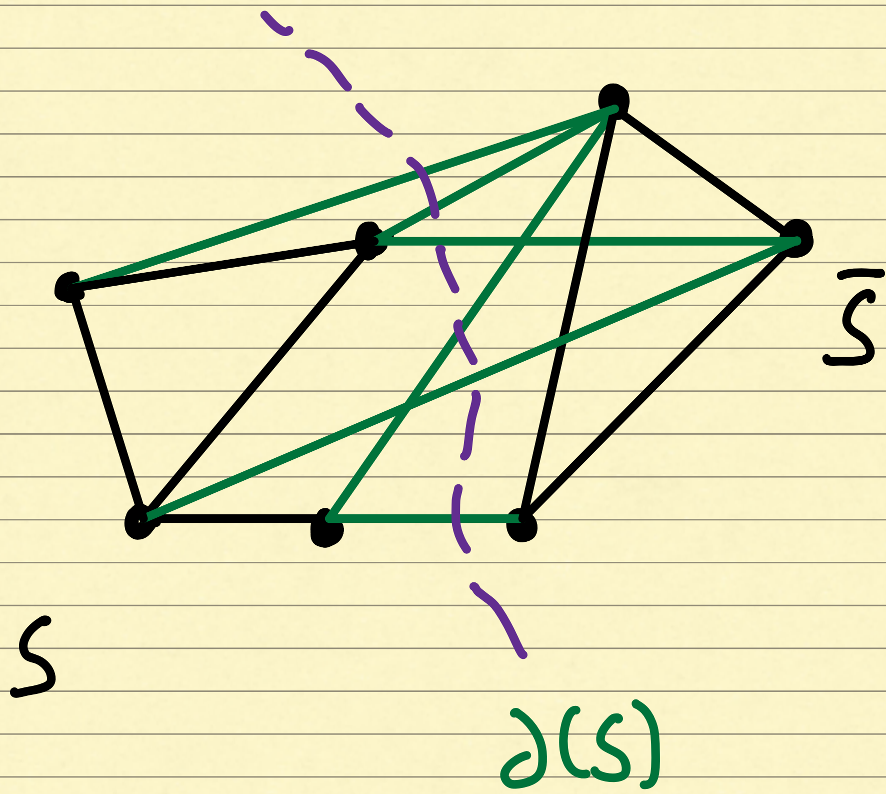
Graph $G = (V, E)$ $|E|$ small



Expanders: Combinatorial Def.

Graph $G=(V,E)$ d-regular n-vtx

"No Small Cuts"



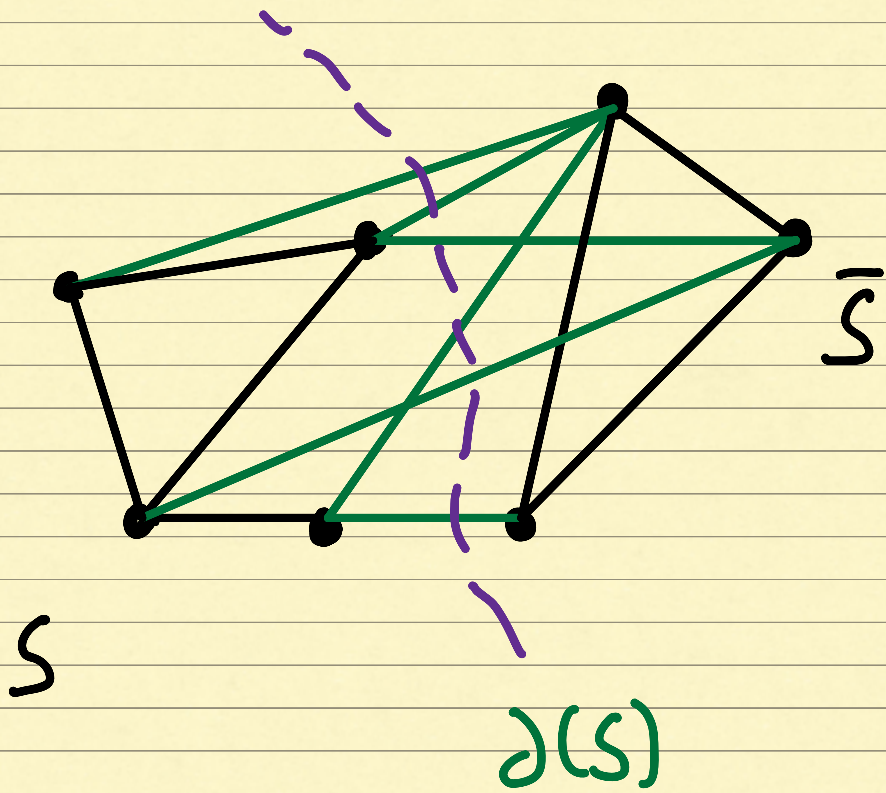
Expanders: Combinatorial Def.

Graph $G=(V,E)$ d-regular n-vtx

"No Small Cuts"

(Edge) Boundary of $S \subseteq V$

$$\partial(S) := \{(s,v) \in E \mid s \in S, v \in \bar{S}\}$$



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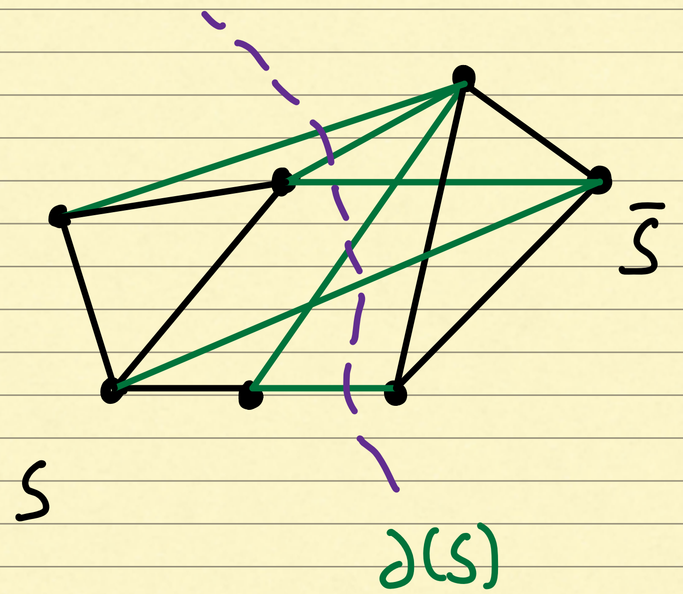


Conductance of S

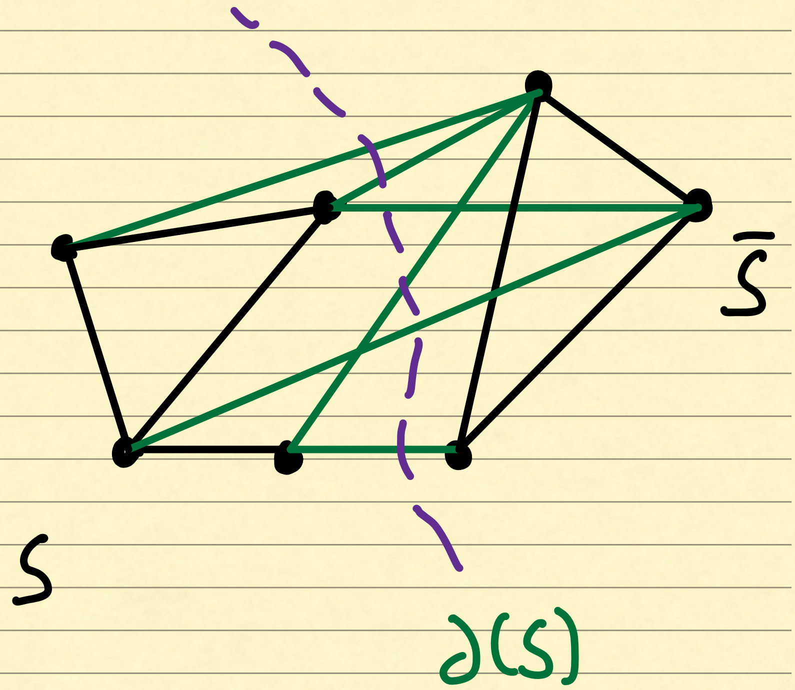
$$\Phi_G(S) := \frac{|\partial(S)|}{d|S|}$$

Expanders: Combinatorial Def.

Graph $G=(V,E)$ d-regular n-vtx

Comb. Expansion

$$\Phi(G) := \min_{\substack{S \subseteq V \\ 1 \le |S| \le \frac{n}{2}}} \Phi_G(S)$$



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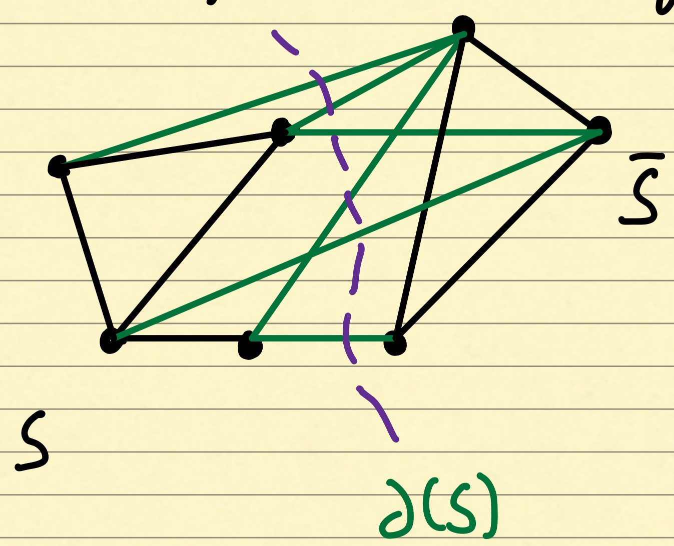

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Want $\Phi(G) \geq \varepsilon > 0$ for a constant ε

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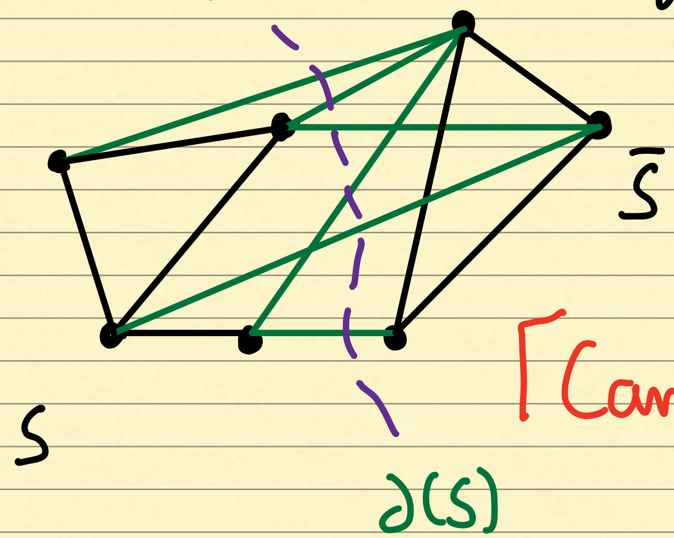

[Can we check this??]

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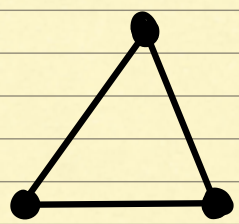
Expanders: Algebraic Def.

Graph $G=(V,E)$ d-regular n-vtx

Adjacency Matrix $A \in \mathbb{R}^{n\times n}$ of G

$$A_{u,v} = \mathbb{1}[u \sim_G v]$$

Ex:

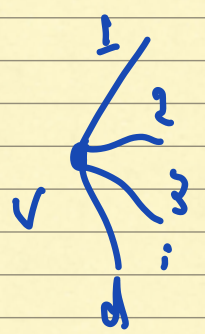


G

$$\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\\1\end{pmatrix}$$

A

$$\vec{1} = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \qquad A\vec{1} = d\,\vec{1}$$



Expanders: Algebraic Def.

Graph $G=(V,E)$ d-regular n-vtx

Adjacency Matrix $A \in \mathbb{R}^{n \times n}$ of G

$$A_{u,v} = \mathbb{1}_{[u \sim_G v]}$$

A is symmetric (over $\mathbb{R}$)

$\Downarrow$ Spectral Theorem 

$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^T$$ with $\{v_i\}_{i \in [n]}$ ONB

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

$$\lambda_1 = d \qquad v_1 = \vec{1}$$

Expanders: Algebraic Def.

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$$A = \sum_{j=1}^{n} \lambda_j v_j v_j^T$$

$$\begin{array}{ccccccc} \lambda_1 & \geq & \lambda_2 & \geq & \cdots & \geq & \lambda_n \\ \parallel & & & & & & \vee\!\!\mid \\ d & & & & & & -d \end{array} \qquad v_1 = \vec{1}/\sqrt{n}$$

Algebraic Expansion

$$\lambda(G) := \max\{\underline{|\lambda_2|}, \underline{|\lambda_n|}\}$$

Want $\underline{\lambda(G)}$ small $\ll d$

Expanders: Algebraic Def.

Graph $G=(V,E)$ d-regular n-vtx

Adjacency Matrix $A \in \mathbb{R}^{n\times n}$ of G

$$A = \sum_{j=1}^{n} \lambda_j v_j v_j^T \qquad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

Algebraic Expansion

$$\lambda(G) := \max\{|\lambda_2|, |\lambda_n|\}$$

Key Algebraic Perspective

$$A = \frac{d}{n} \underbrace{\vec{1}\vec{1}^T}_{\text{adj. matrix of complete graph with self loops}} + \overbrace{\lambda E}^{\text{error term}} \qquad \|E\|_{op} \leq 1$$

Expanders: Algebraic Def.

Graph $G=(V,E)$ d-regular n-vtx

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$\|E\|_{op} \leq 1$

$$A \simeq \frac{d}{n} J$$

complete graph with density d/n

Expander Definitions

Equivalent Views

Combinatorial: $\Phi(G)$ large

$\Updownarrow$ Cheeger's Ineq.

Algebraic: $\lambda(G)$ small

$\Updownarrow$

Random Walk: Fast Mixing

Warm up: Expander Mixing Lemma

Graph $G=(V,E)$ $\quad (n,d,\lambda)$-graph

Suppose we want to compute (approx) $|E(S,\bar{S})|$?

$$|E(S,\bar{S})| = \mathbb{1}_S^T A \mathbb{1}_{\bar{S}}$$

Warm up: Expander Mixing Lemma

Graph $G=(V,E)$ $\quad (n,d,\lambda)$

Suppose we want to compute $|E(S,\bar{S})|$?
(approx)

$$|E(S,\bar{S})| = 1_S^{t} A 1_{\bar{S}}$$

Recall that $A = \frac{d}{n} J + \lambda E$

$$\left(A \simeq \frac{d}{n} J \right)$$

$$|E(S,\bar{S})| \approx \frac{d}{n} 1_S^{t} J 1_{\bar{S}} = \frac{d}{n} |S| |\bar{S}|$$

Warm up: Expander Mixing Lemma

Graph $G=(V,E)$ (n,d,λ)

$$|E(S,\bar{S})| = 1_S^T A 1_{\bar{S}}$$

Recall that $A = \frac{d}{n} J + \lambda E$

$$\left(A \approx \frac{d}{n} J \right)$$

$$|E(S,\bar{S})| \approx \frac{d}{n} |S||\bar{S}|$$

$\|E\|_{op} \leq 1$

More rigorously,

$$|E(S,\bar{S})| = \frac{d}{n} |S||\bar{S}| + \underbrace{\lambda 1_S^T E 1_{\bar{S}}}_{\pm \lambda \sqrt{|S||\bar{S}|}}$$

error term

Some Applications

- Coding Theory
- Complexity Theory
- Hardness of Approximation

  ⋮

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- Coding Theory
- Complexity Theory
- Hardness of Approximation
- Pseudorandomness
- Sampling and Counting
- Algorithm Design
- Property Testing
- Metric Spaces
- Group Theory
- Number Theory

$\vdots$

Some Applications

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⋮

- Your new application

Some Applications

(an appetizer)

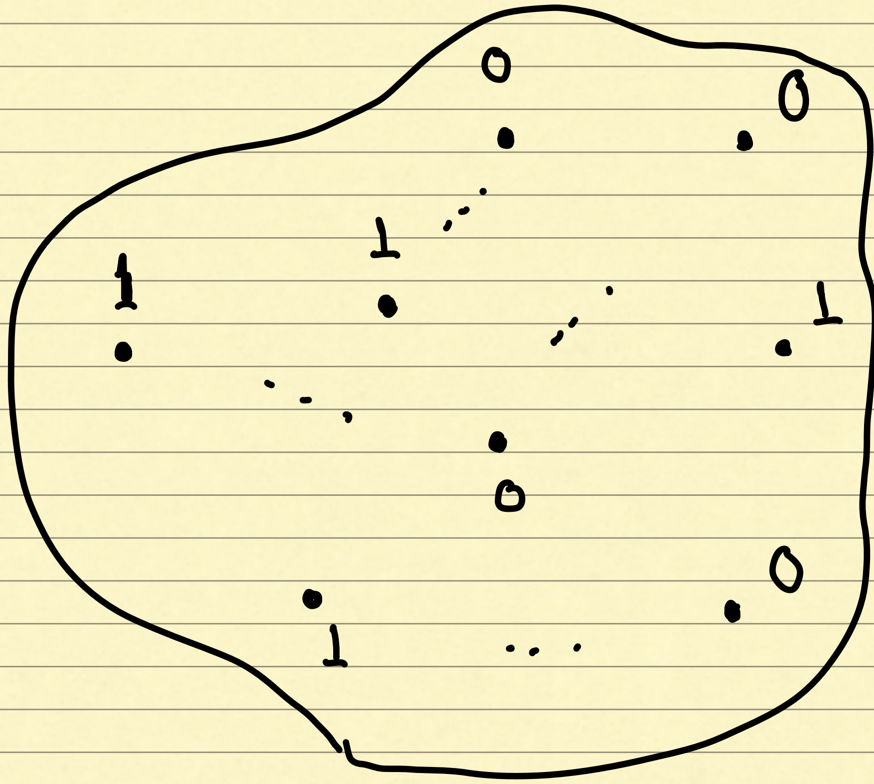
- Derandomization
- Complexity Theory

How does expansion appear?

Chernoff Bound

n 0/1 values

$$\mu := \frac{\#\text{ of } 1\text{'s}}{n}$$

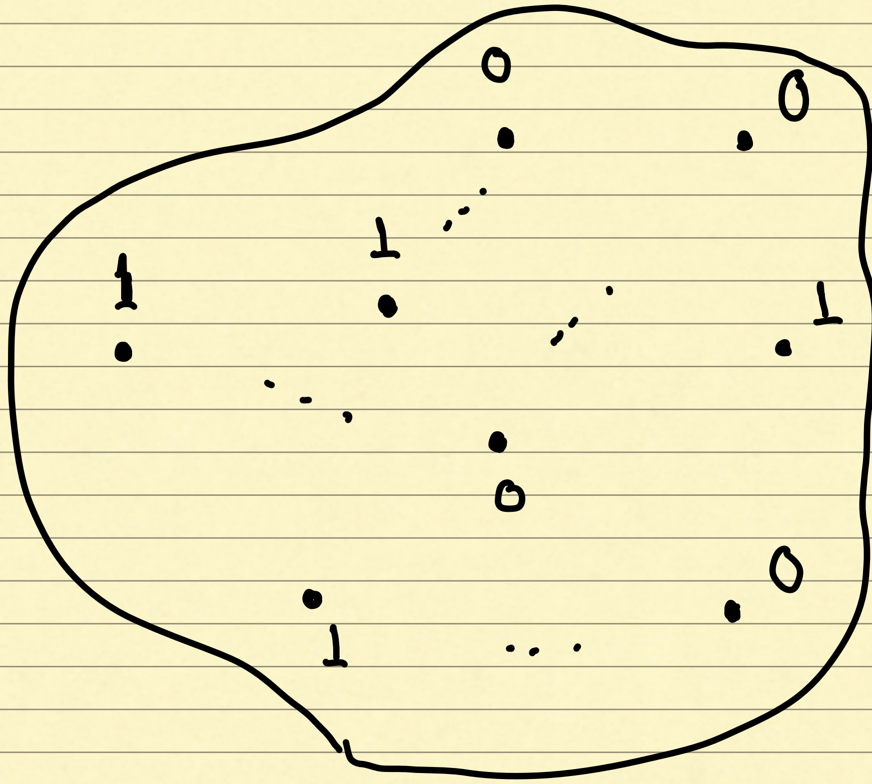


Sample k indepent values uniformly

$X_1, \dots, X_k$

Chernoff Bound

n 0/1 values



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Sample k indepent values uniformly

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$$\Pr\left[\left|\frac{1}{k}\sum X_i - \mu\right| > \varepsilon\right] \leq \exp\left(-\Omega\left(\varepsilon^2 k\right)\right)$$

Chernoff Bound

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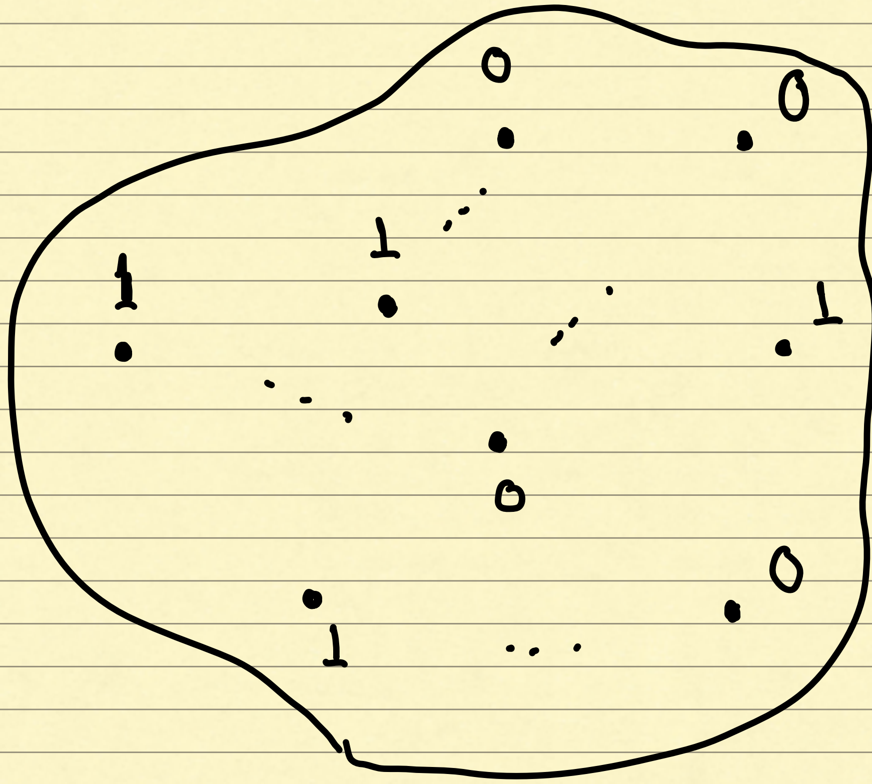

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$\#$ of random bits $\approx k \cdot \log_2(n)$

Expander Chernoff Bound

$G = (V = [n], E)$

d-regular $\quad d = O(1)$

n 0/1 values

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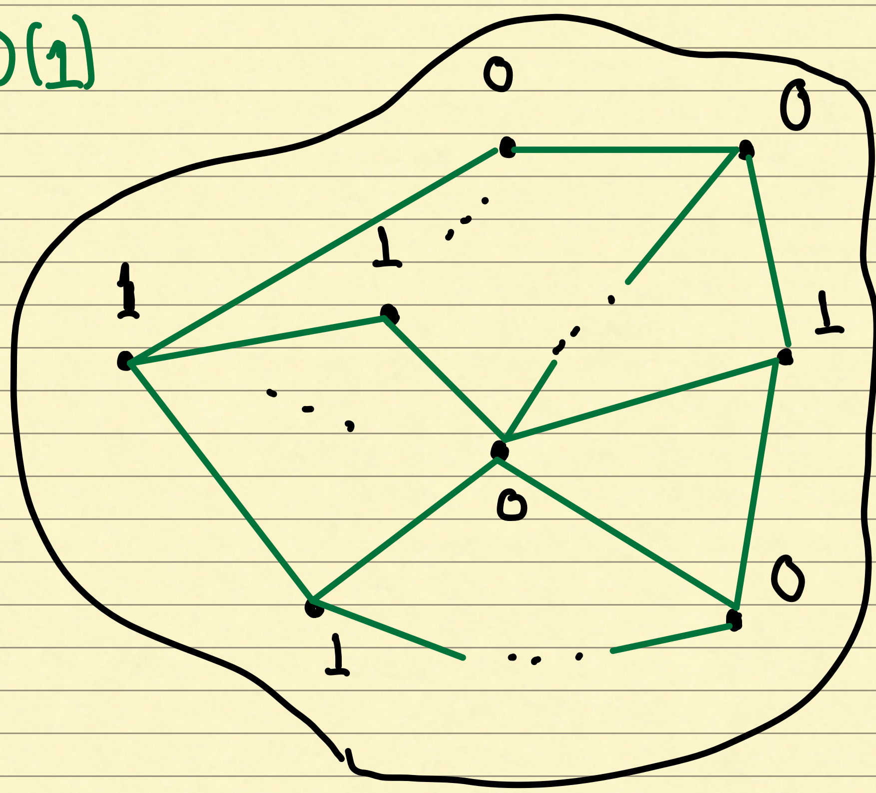

Expander Chernoff Bound

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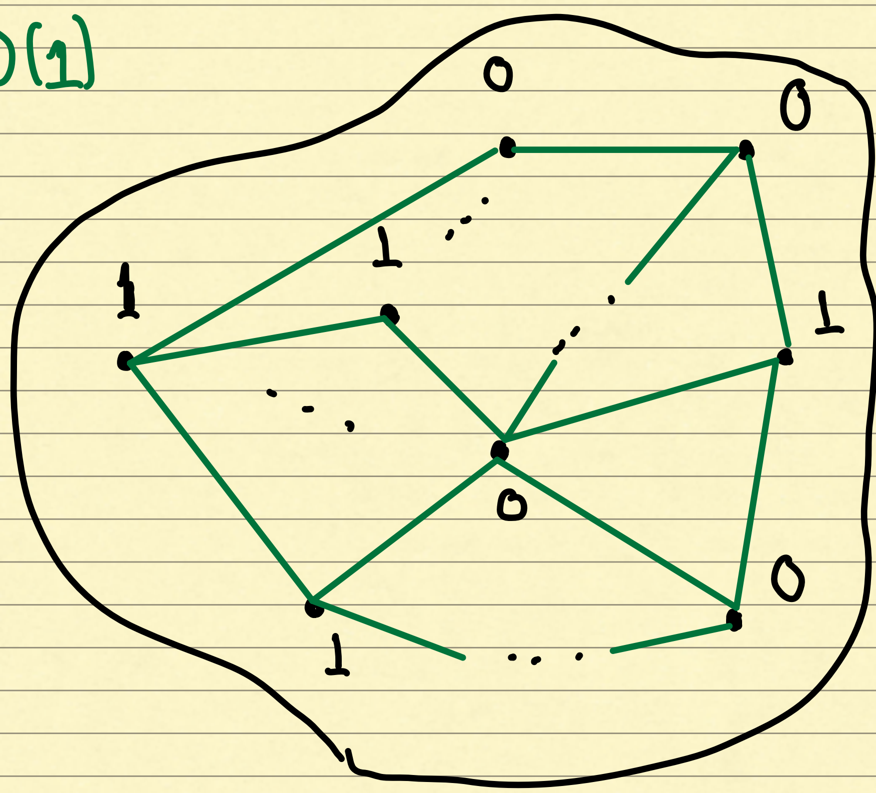


$\mu := \dfrac{\#\text{ of } 1\text{'s}}{n}$

Sample k values along a random walk

$X_1, \ldots, X_k$ $\qquad$ on G

Expander Chernoff Bound

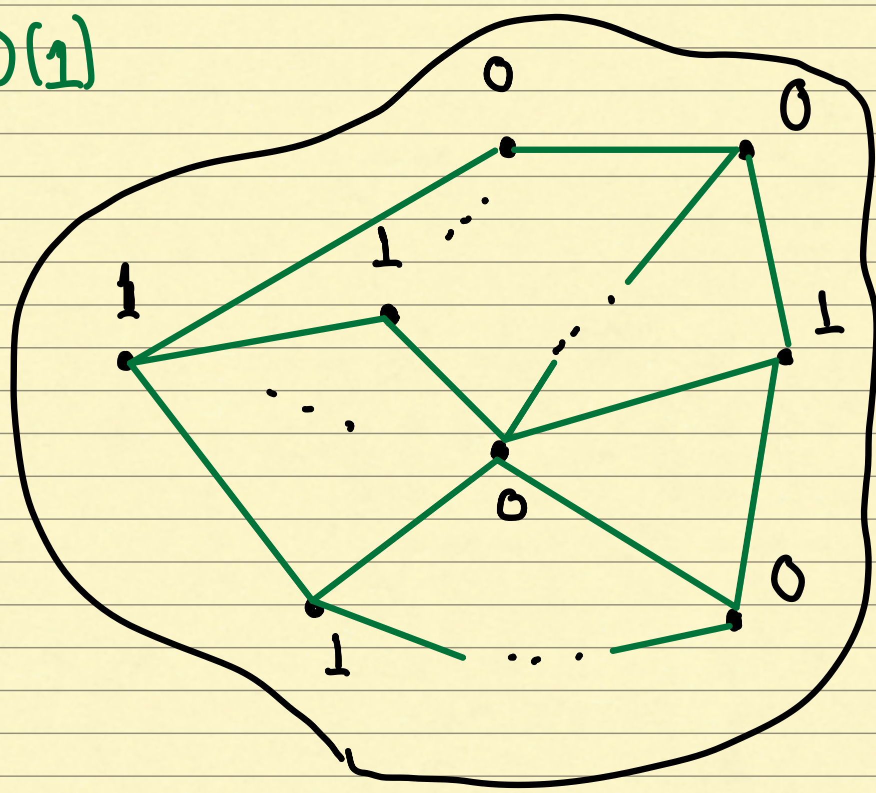
$G = (V = [n], E)$

d-regular $\quad d = O(1)$

$\boxed{\lambda(G) \leq \lambda}$

$\mu := \dfrac{\#\text{ of } 1\text{'s}}{n}$

n 0/1 values



Sample k values along a random walk on G

$X_1, \ldots, X_k$

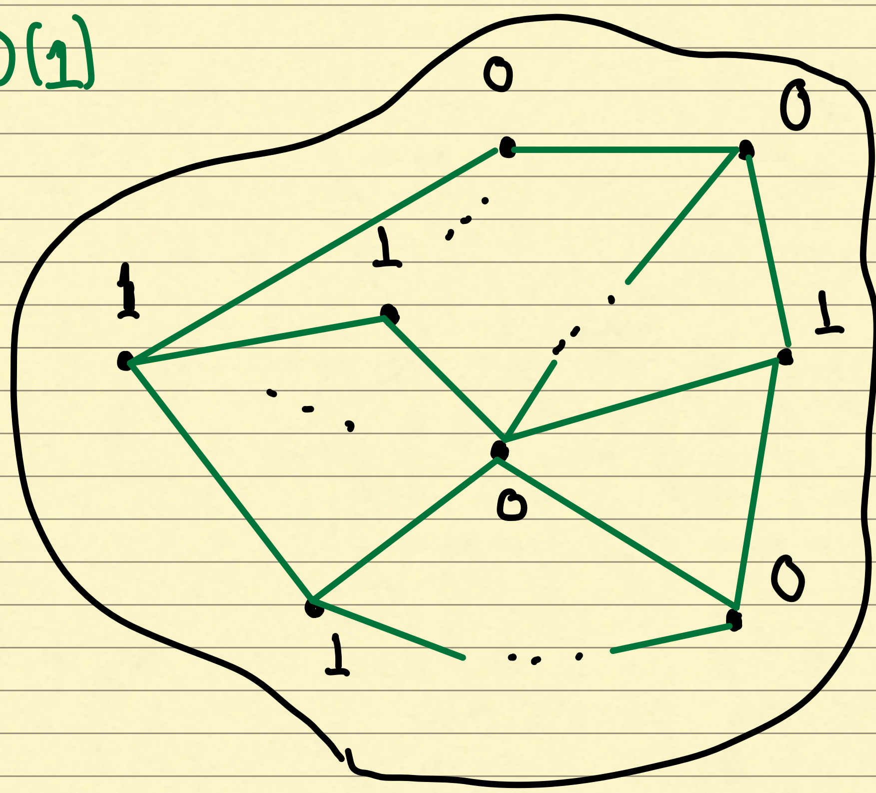
Expander Chernoff Bound

$G = (V=[n], E)$ n 0/1 values

d-regular $d = O(1)$

$\boxed{\lambda(G) \le \lambda}$

$\mu := \dfrac{\#\text{ of } 1\text{'s}}{n}$



Sample k values along a random walk

$X_1, \dots, X_k$ on G

Theorem [Gillman'93]

$$\Pr\left[\left|\frac{1}{k}\sum X_i - \mu\right| > \varepsilon\right] \le \exp\left(-\Omega_\lambda(\varepsilon^2 k)\right)$$

\# of random bits $\approx \cancel{k}\cdot\log_2(n) + O(k)$

Complexity: $SL = L$

Input: $G=(V,E)$ an n-vtx graph,

$s,t \in V$.

Question: $\exists$ s-t path in G?

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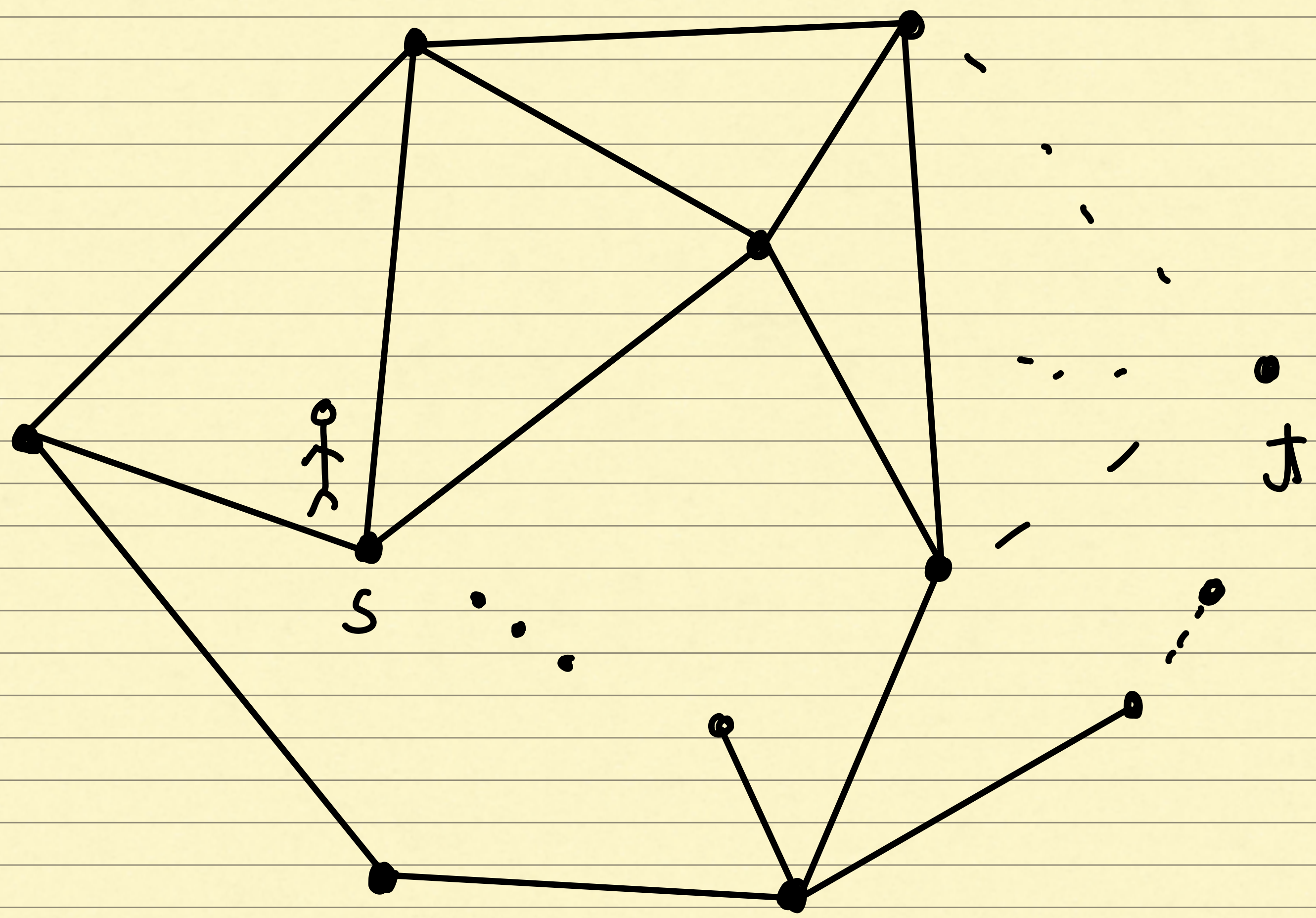

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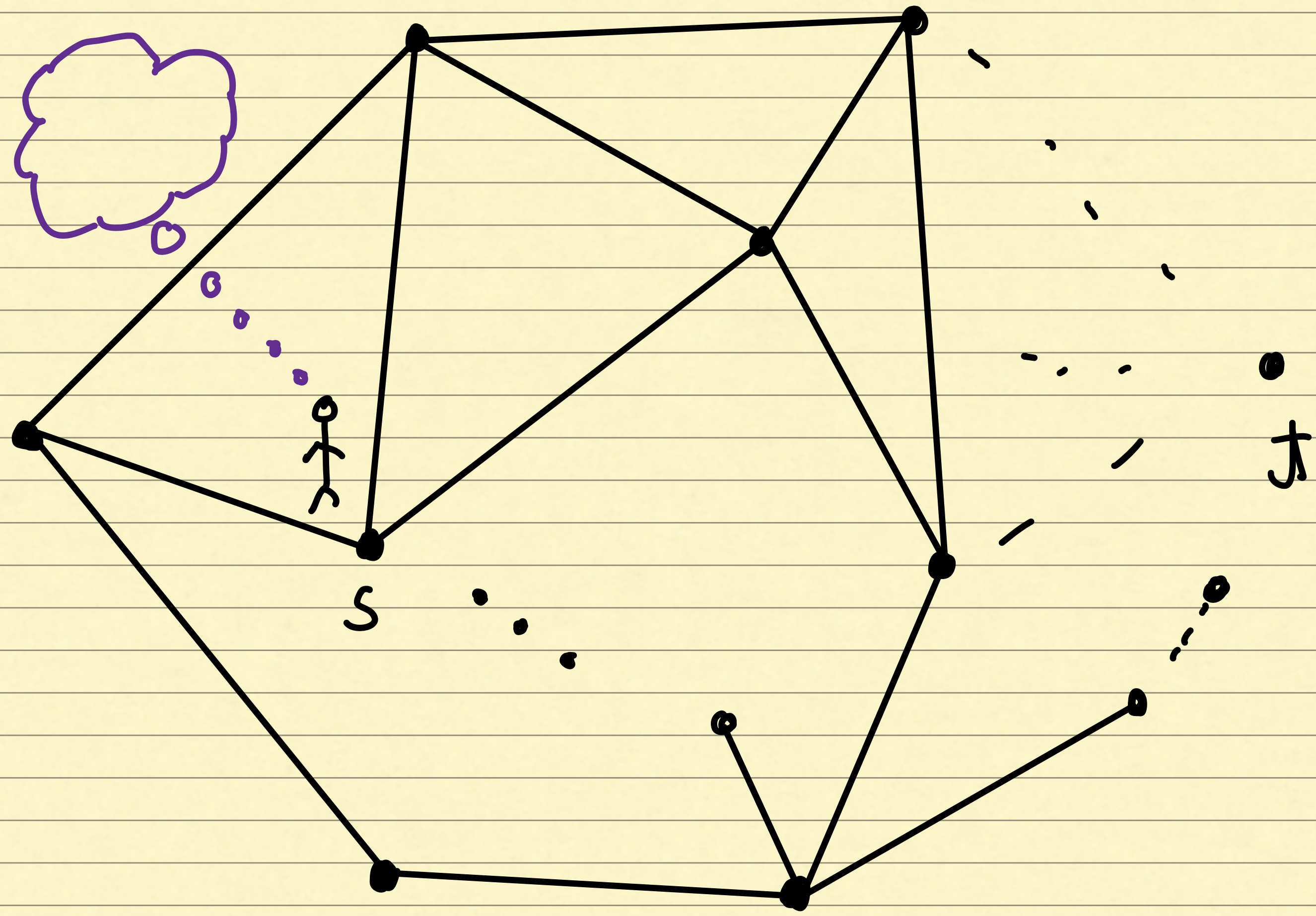
We can only remember $O(1)$ vertices



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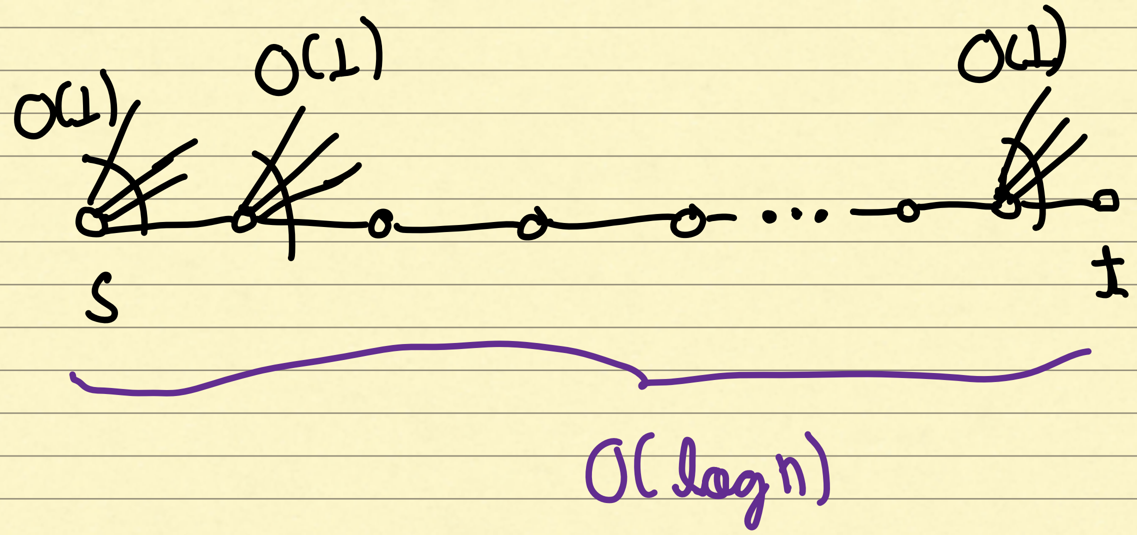
Observation: This is easy if G is a constant degree expander
(diameter is $O(\log n)$)



Complexity: SL = L

Input: $G=(V,E)$ an n-vtx graph, $s,t \in V$.

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This can be done!

Transform (connected comp.) of G into expander!

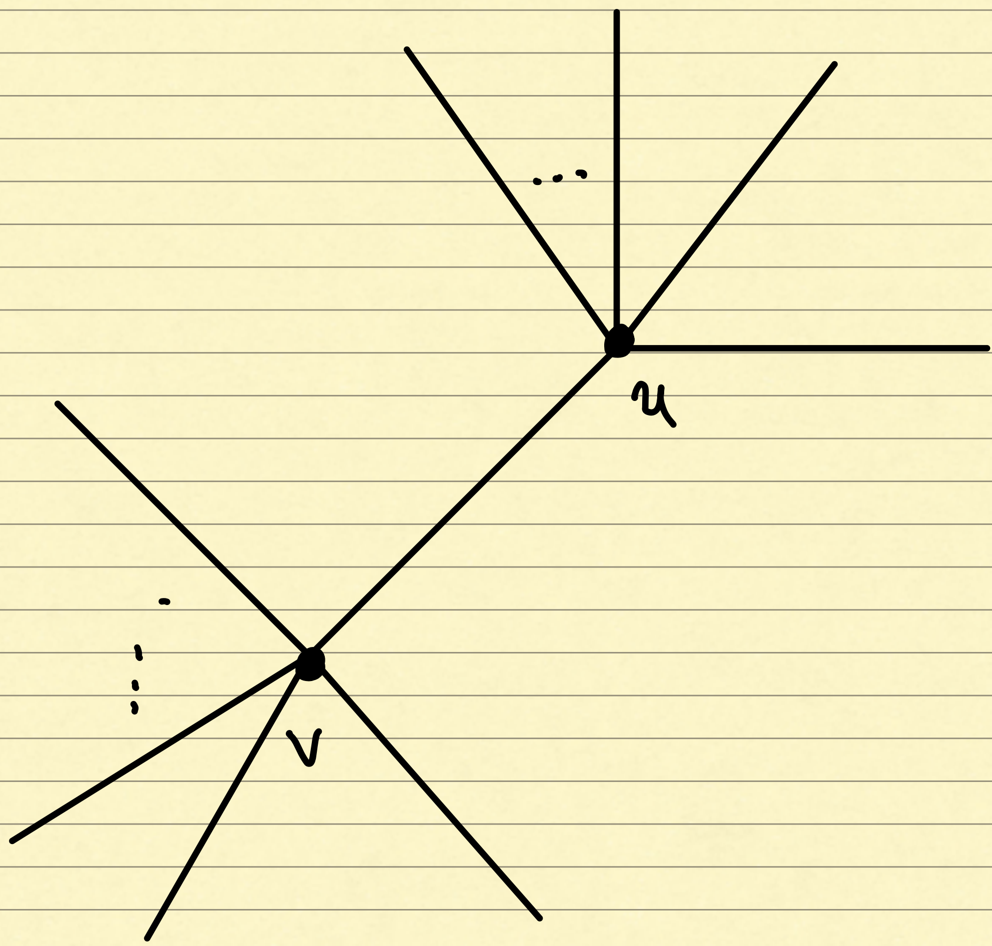
Thm [Reingold'05] SL = L

Key Technique: Zig-Zag product

Towards Zig-Zag

First a simpler goal: Degree Reduction

Let G be n-vtx degree D ("large")



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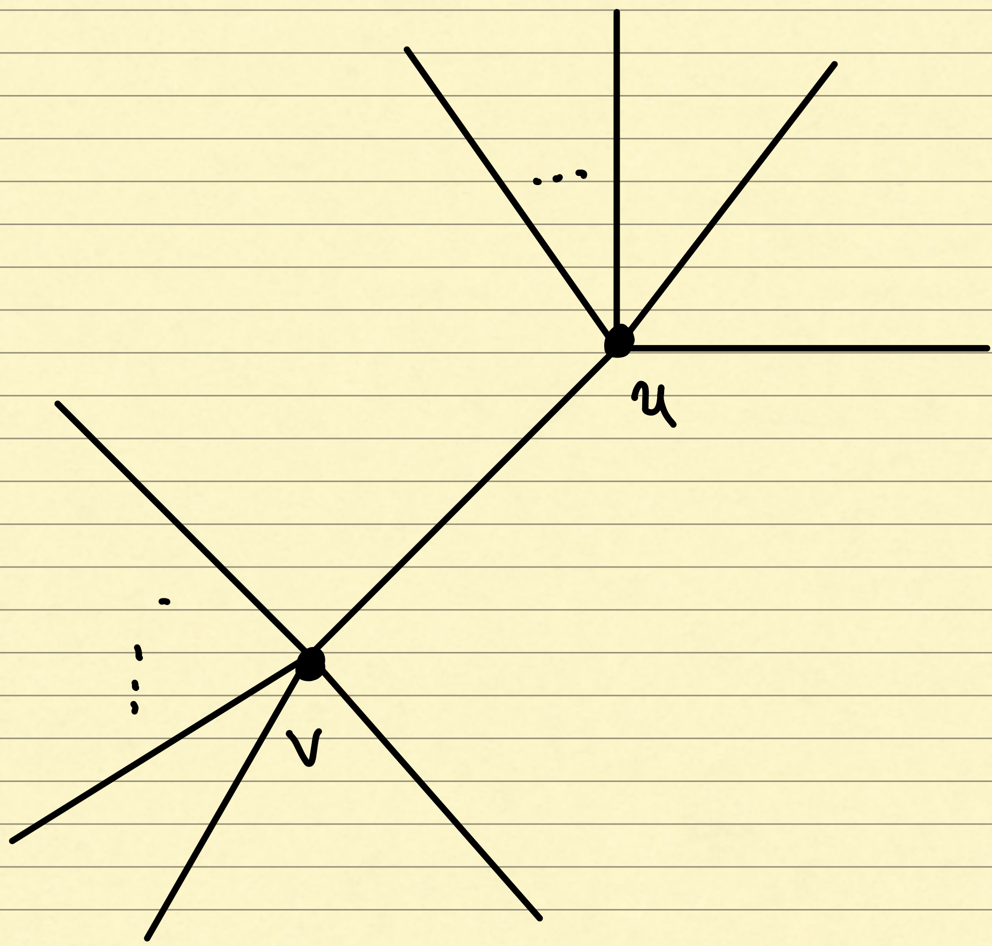

Hint: use a D-vtx graph H of degree $d \ll D$

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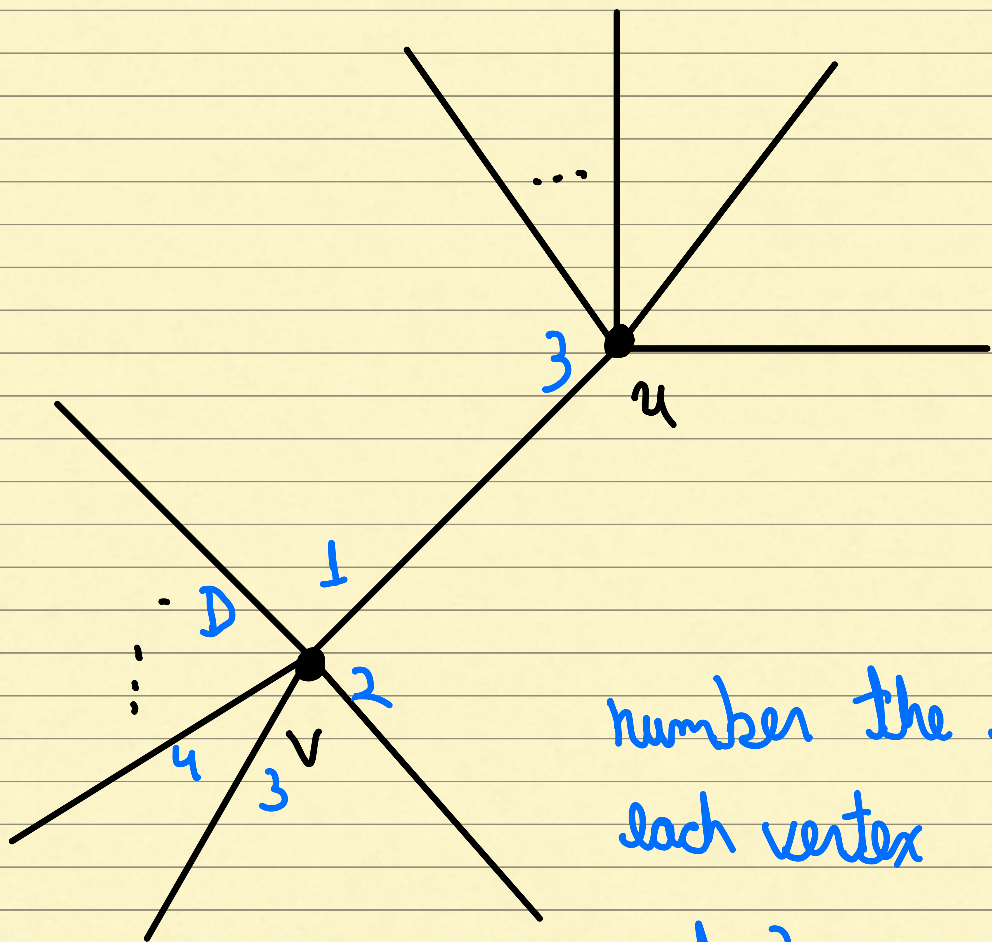


number the edges of each vertex using $1, 2, ..., D$

Hint: use a D-vtx graph H of degree $d \ll D$

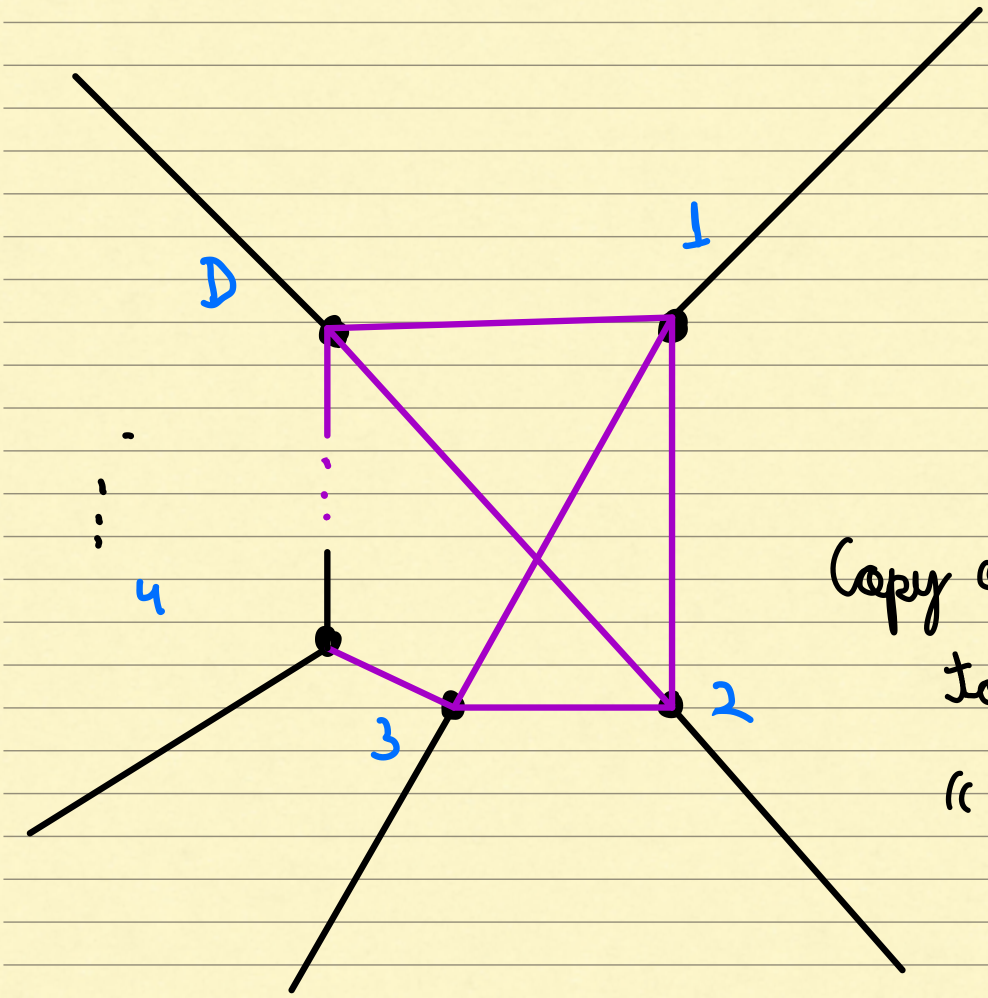
$$H = (V = \underline{[D]}, E)$$

Towards Zig-Zag

First a simpler goal: Degree Reduction

Let G be n-vtx degree D

Replace each vertex of G by a copy of H



Copy of H associated to v

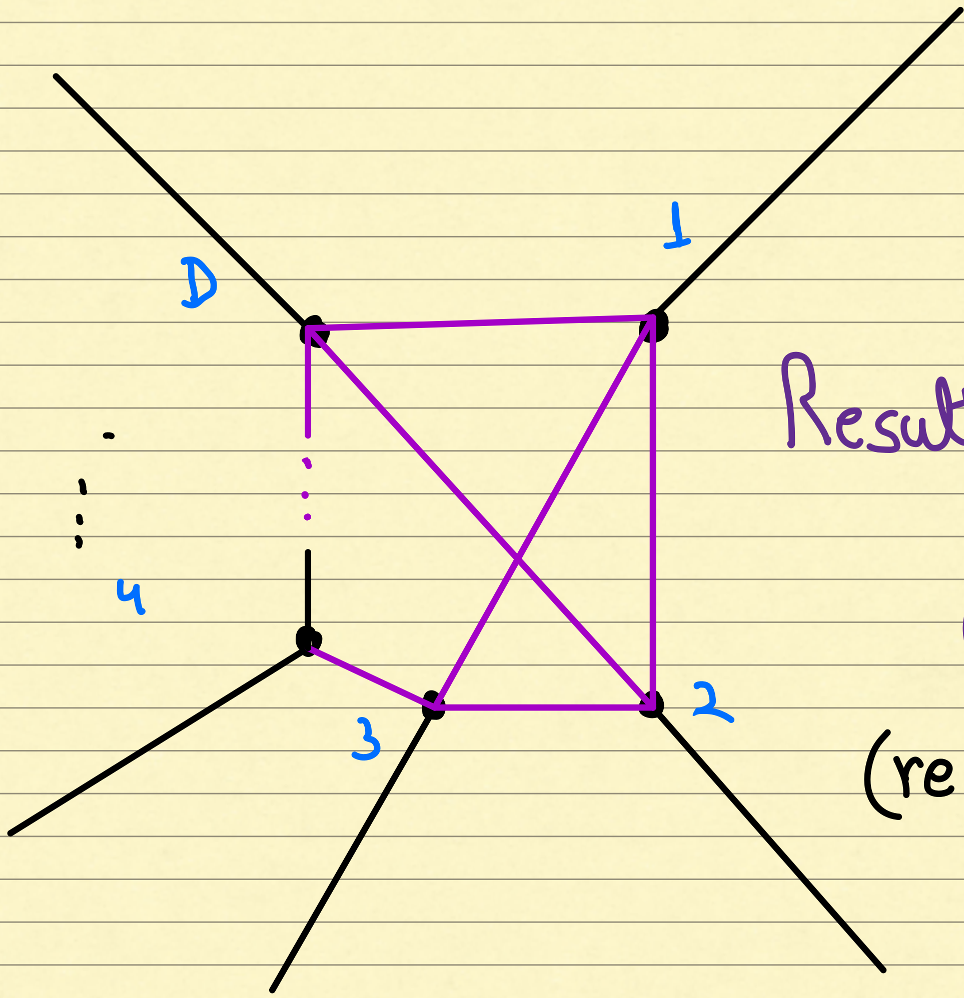
"cloud"

$H = (V = [D], E)$ degree $d \ll D$

Towards Zig-Zag

First a simpler goal: Degree Reduction

Let G be n-vtx degree D



Replace each vertex of G by a copy of H

Resulting Graph:

$$G \text{ (r) } H$$

(replacement product)

deg: $d+1$ ✓

$H = (V = [D], E)$ degree $d \ll D$

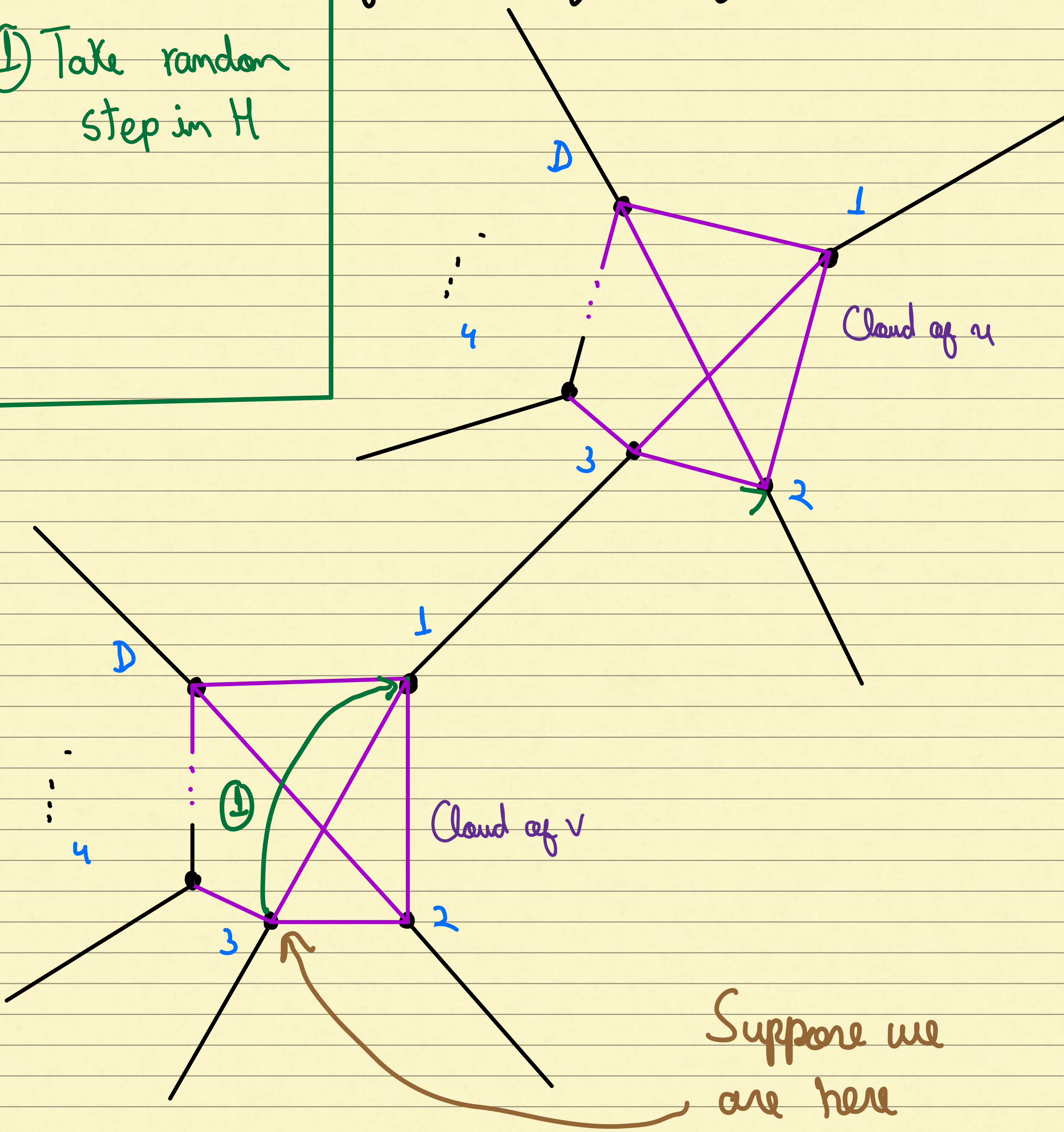
Zig-Zag Product

$G \text{ ⓩ } H$ is obtained from $G \text{ ⓡ } H$ by a "zig-zag walk"

Zig-Zag Product

$G \circledz H$ is obtained from $G \circledr H$ by a "zig-zag walk"

① Take random step in H

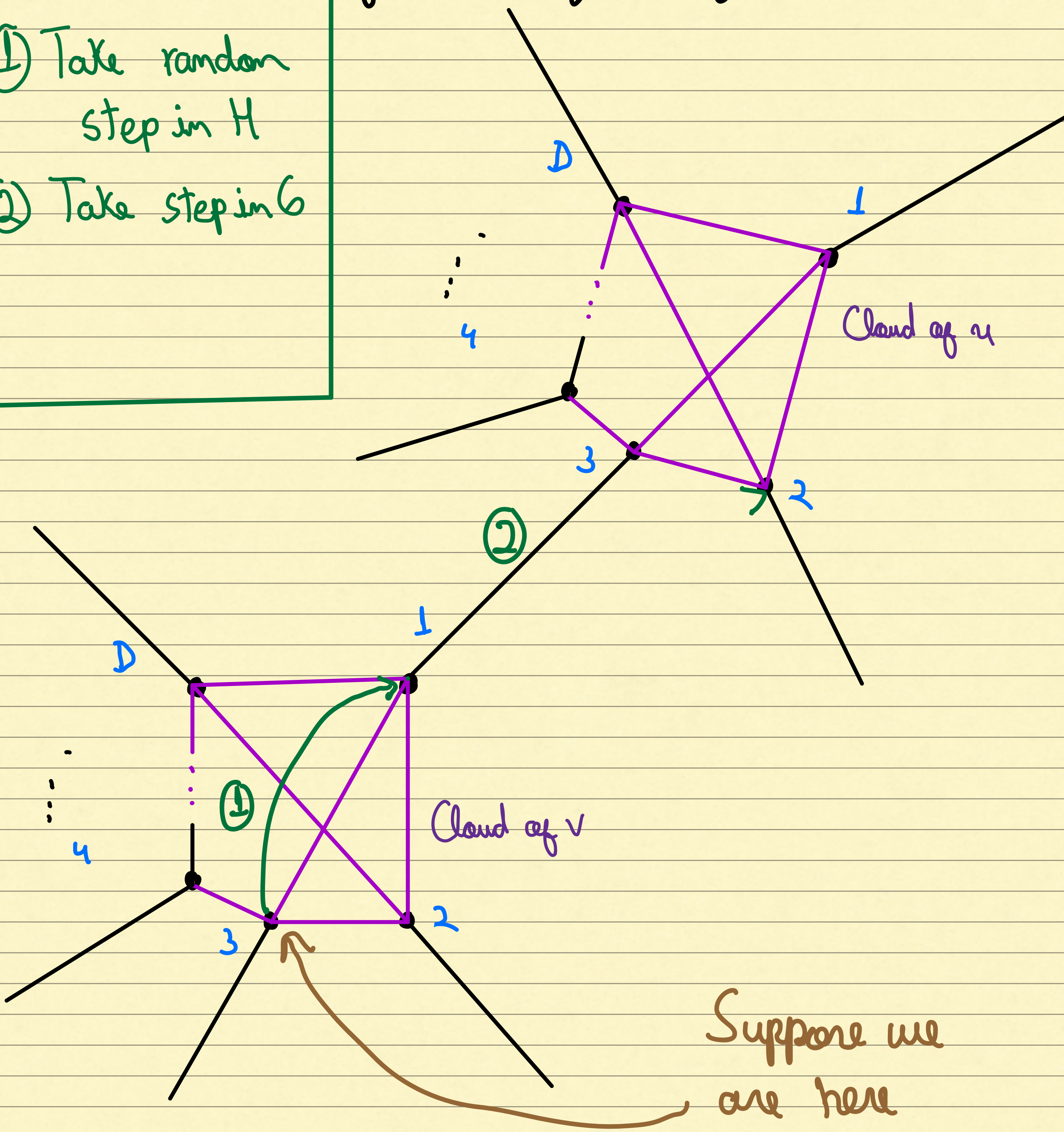


Zig-Zag Product

$G \circledz H$ is obtained from $G \circledr H$ by a "zig-zag walk"

① Take random step in H

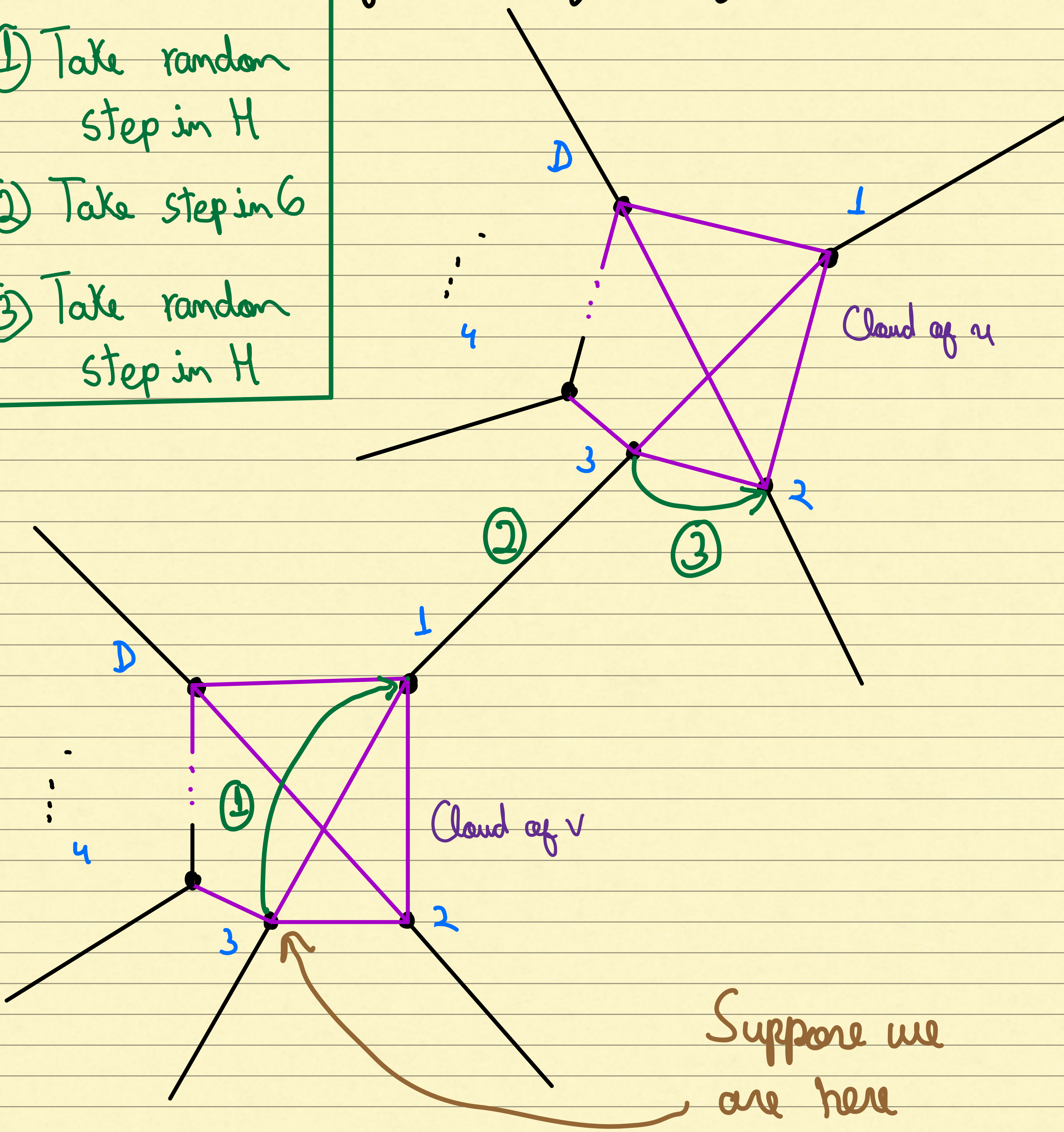
② Take step in G



Zig-Zag Product

G ⓩ H is obtained from G ⓡ H by a "zig-zag walk"

1. Take random step in H
2. Take step in G
3. Take random step in H



Zig-Zag Product

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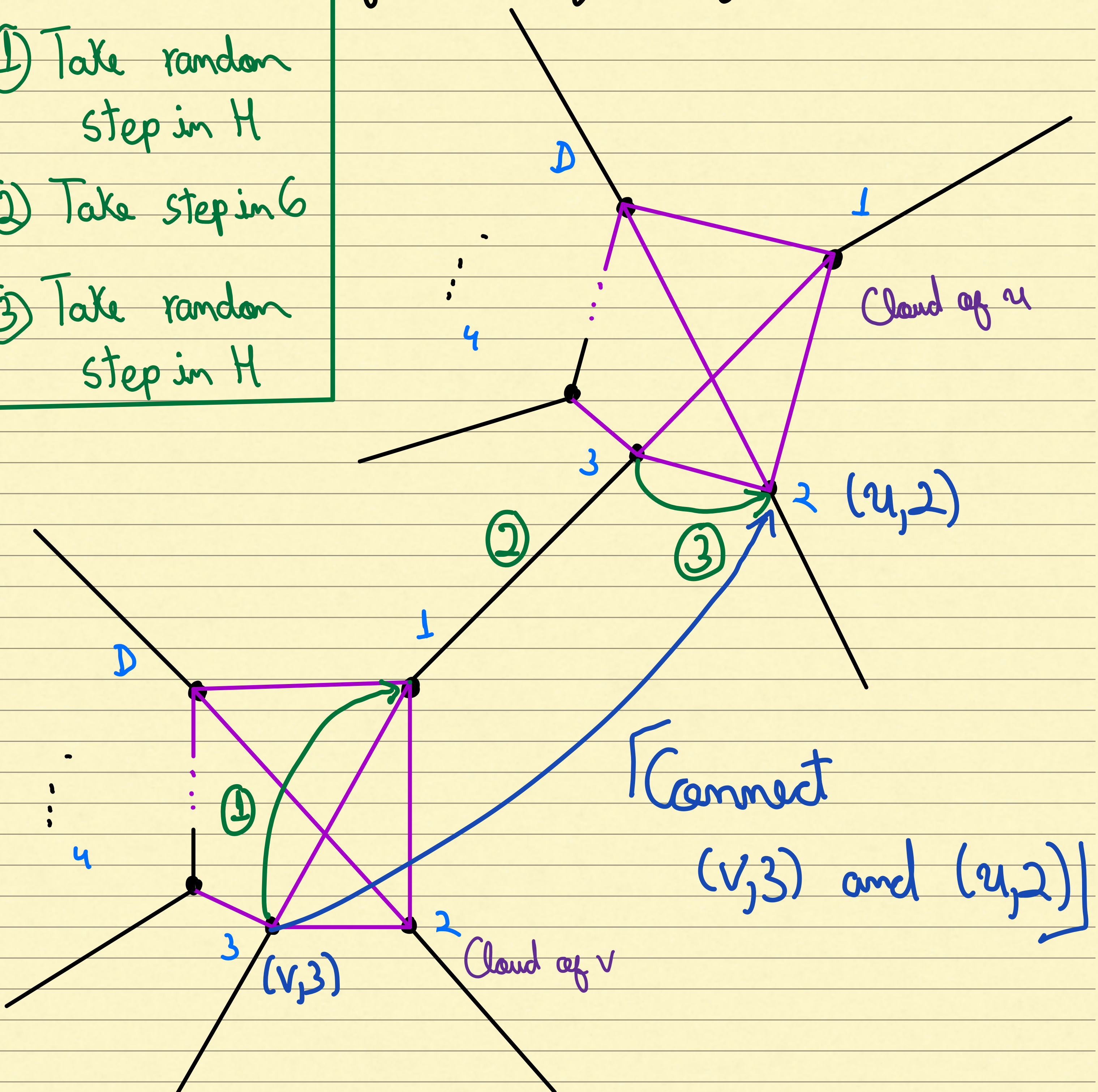


Zig-Zag Product

Theorem [Reingold-Vadhan-Wigderson '00]

If G is (n, D, λ_1)-graph and

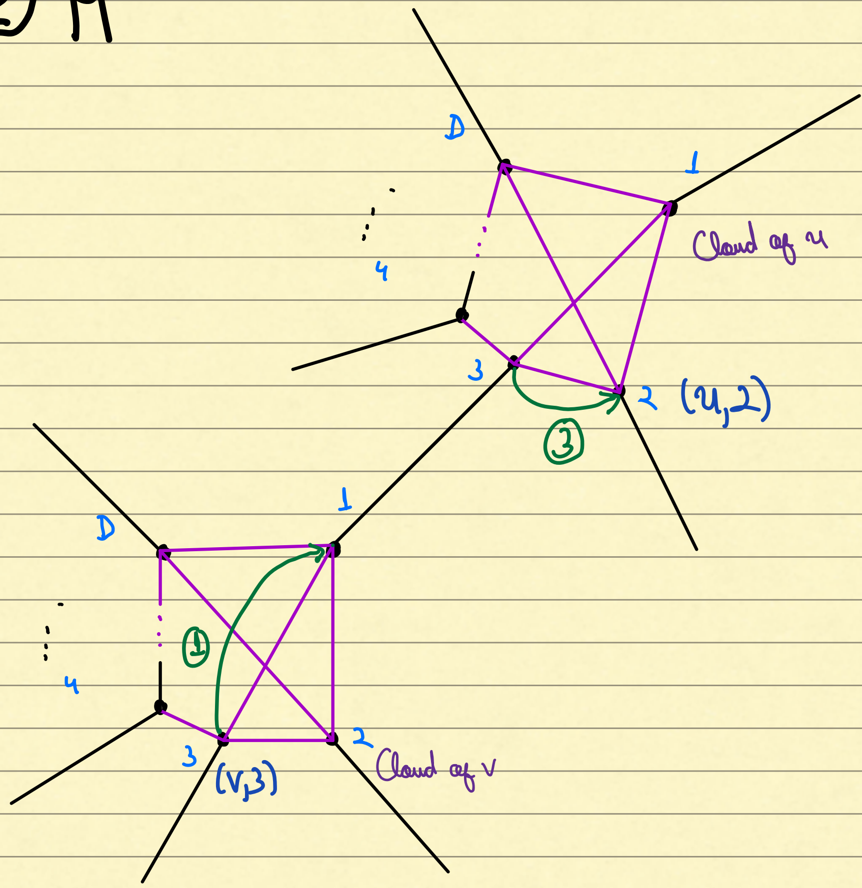
H is (D, d, λ_2)-graph, then

$G \textcircled{z} H$ is $(nD, d^2, f(\lambda_1, \lambda_2))$-graph

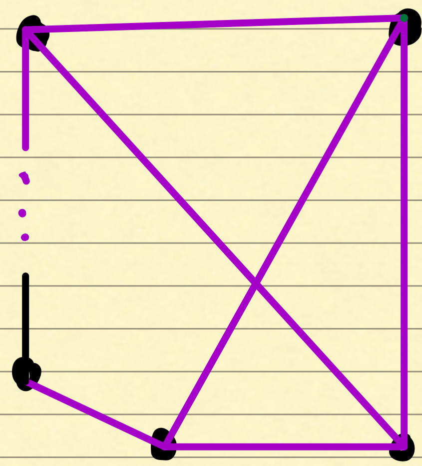
(We will bound $f(\lambda_1, \lambda_2)$ soon)

Zig-Zag Product

$G \text{ ⓩ } H$

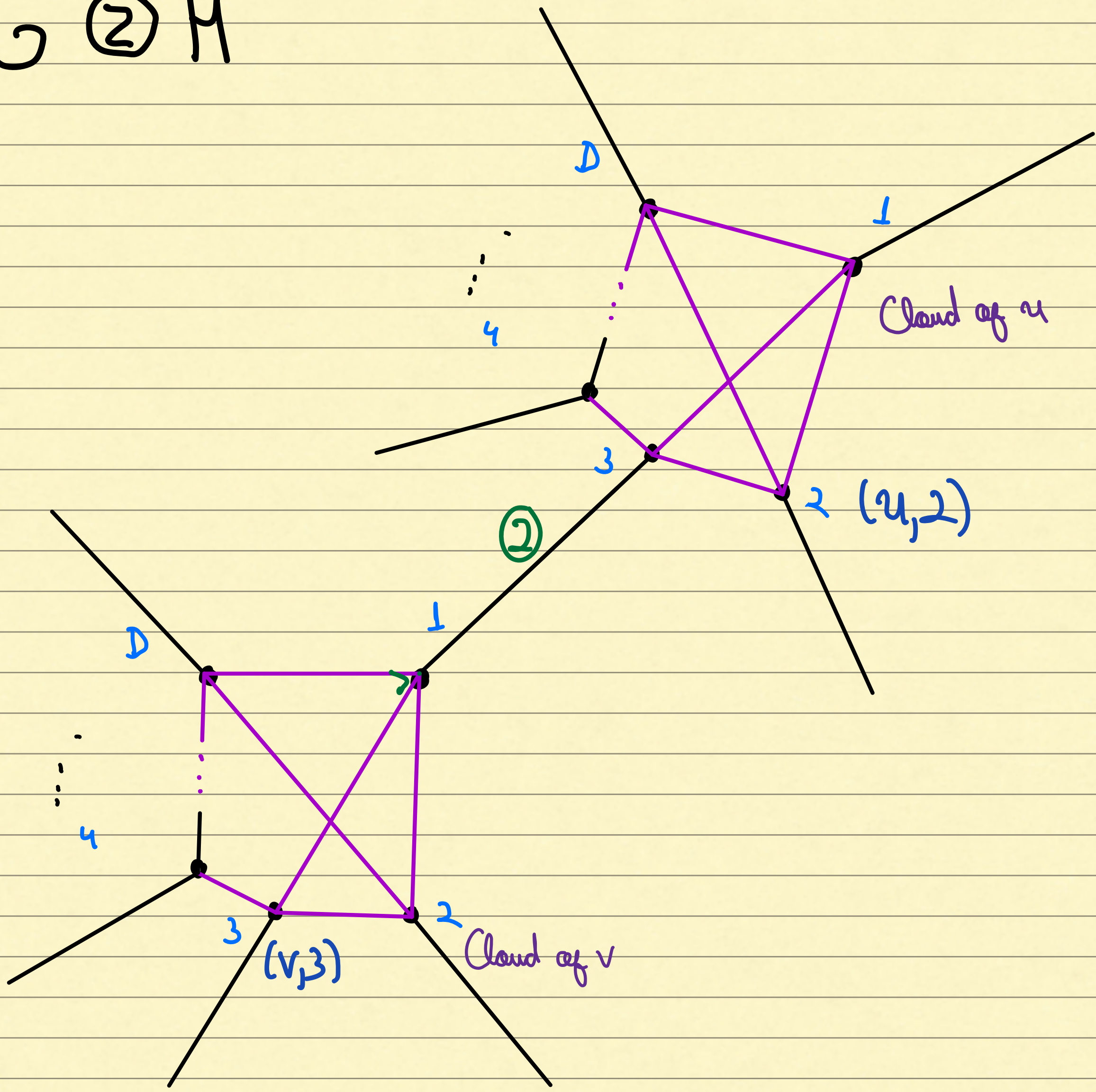


Let A_H be the normalized adj. matrix of H



Zig-Zag Product

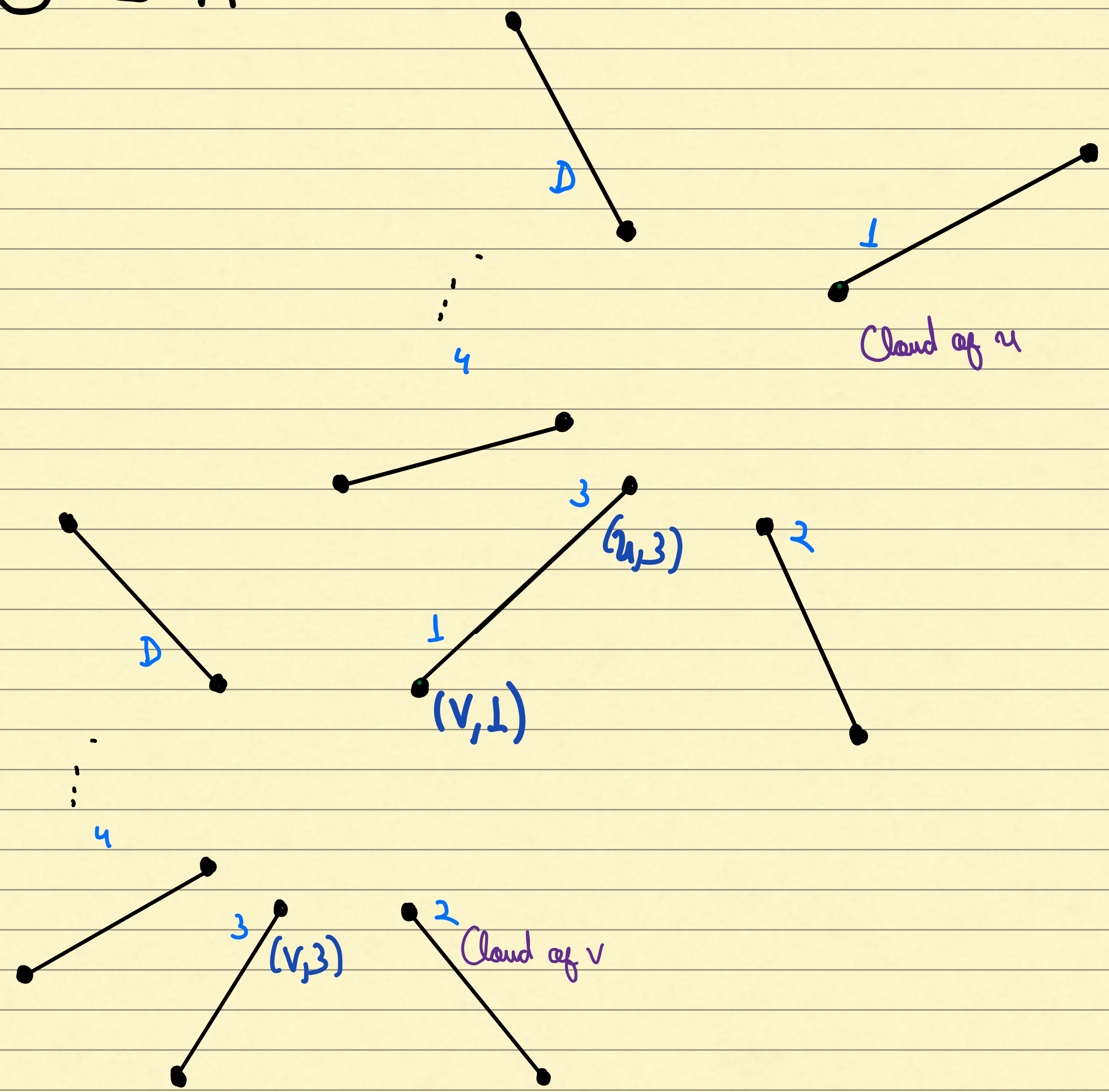
$G \textcircled{z} H$



$$\boxed{\text{Let } \tilde{G} \text{ be the matrix of the action of } G \text{ on } \mathbb{R}^{V(G)\times V(H)}}$$

Zig-Zag Product

$G \, ⓩ \, H$

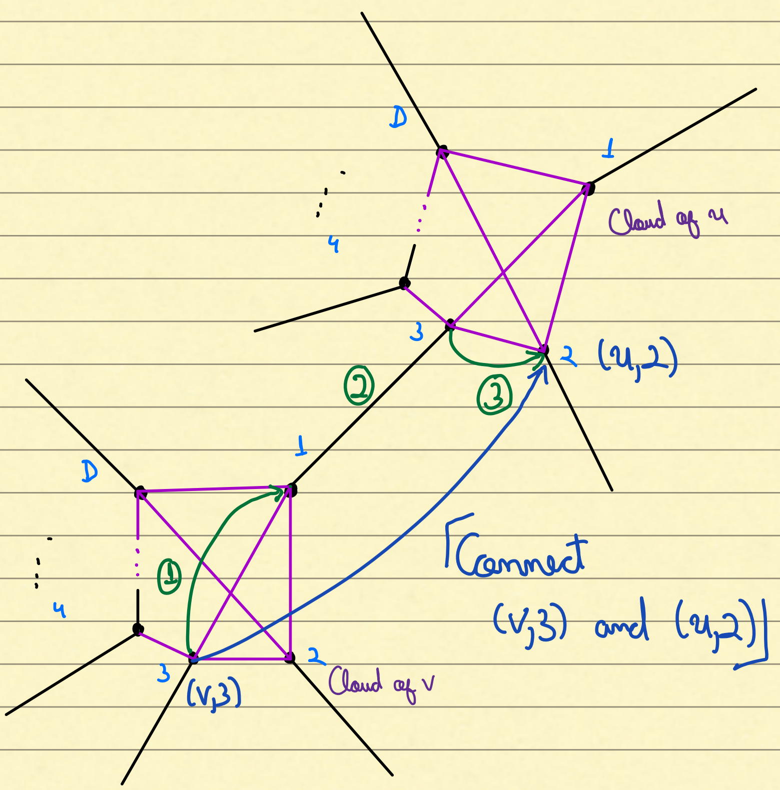


Let $\tilde{G}$ be the matrix of the action of G on $\mathbb{R}^{V(G) \times V(H)}$

Zig-Zag Product

$G \text{ ⓩ } H$

	Operators
① Take random step in H	$I \otimes A_H$
② Take step in G	$\tilde{G}$
③ Take random step in H	$I \otimes A_H$



Let A_H be the normalized adj. matrix of H

Let $\tilde{G}$ be the matrix of the action of G on $\mathbb{R}^{V(G) \times V(H)}$

Zig-Zag Product

$G \textcircled{z} H$

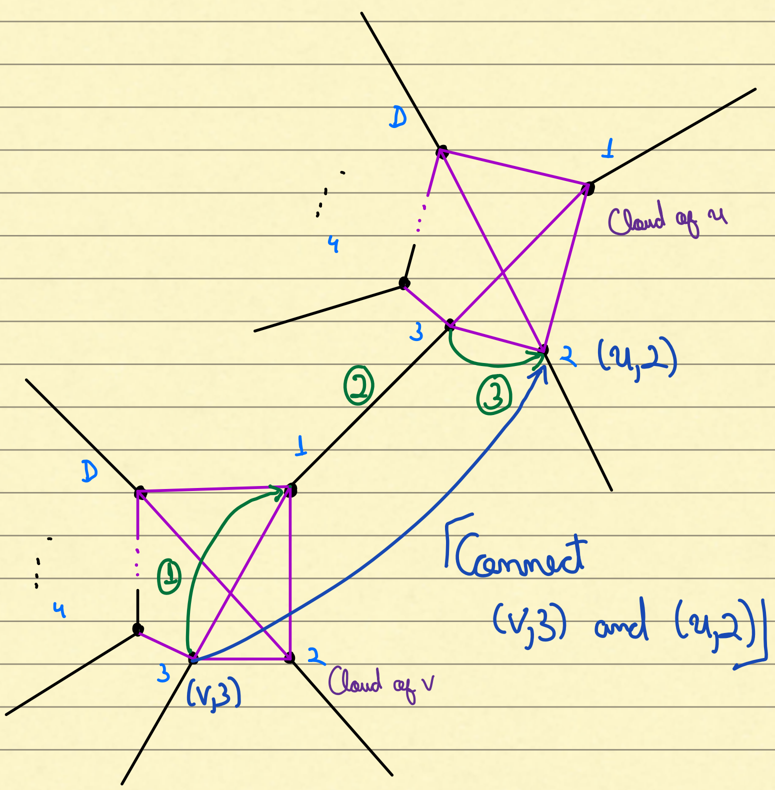
① Take random step in H — $I \otimes A_H$

② Take step in G — $\tilde{G}$

③ Take random step in H — $I \otimes A_H$

Adj. mat. $G \textcircled{z} H$

$$(I \otimes A_H)\tilde{G}(I \otimes A_H)$$



Let A_H be the normalized adj. matrix of H

Let $\tilde{G}$ be the matrix of the action of G on $\mathbb{R}^{V(G)\times V(H)}$

Zig-Zag Product

Special Case : $G \textcircled{z} H$ with H complete graph with self loops

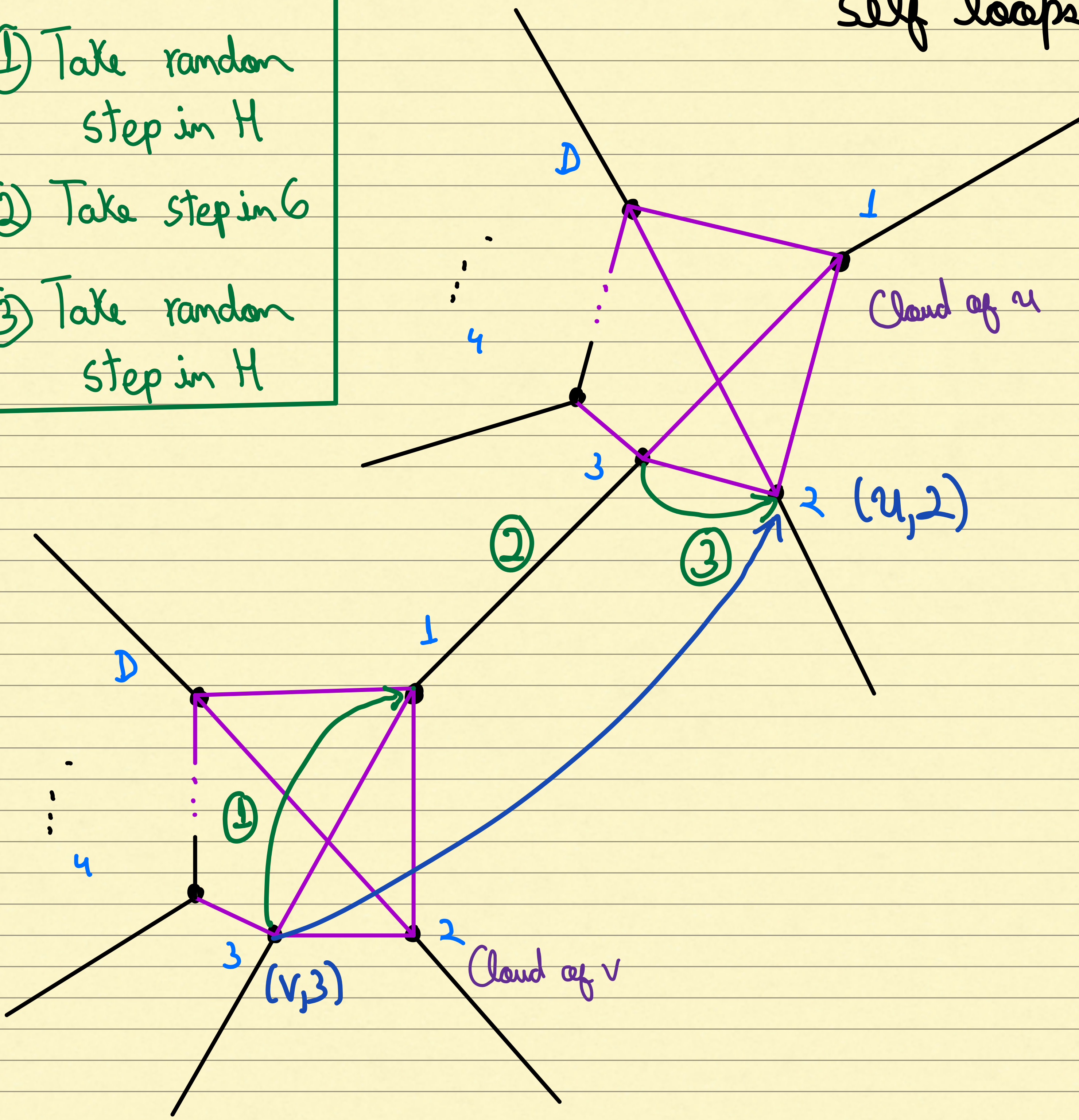
Zig-Zag Product

Special Case: G ⓩ H with H complete graph with self loops

① Take random step in H

② Take step in G

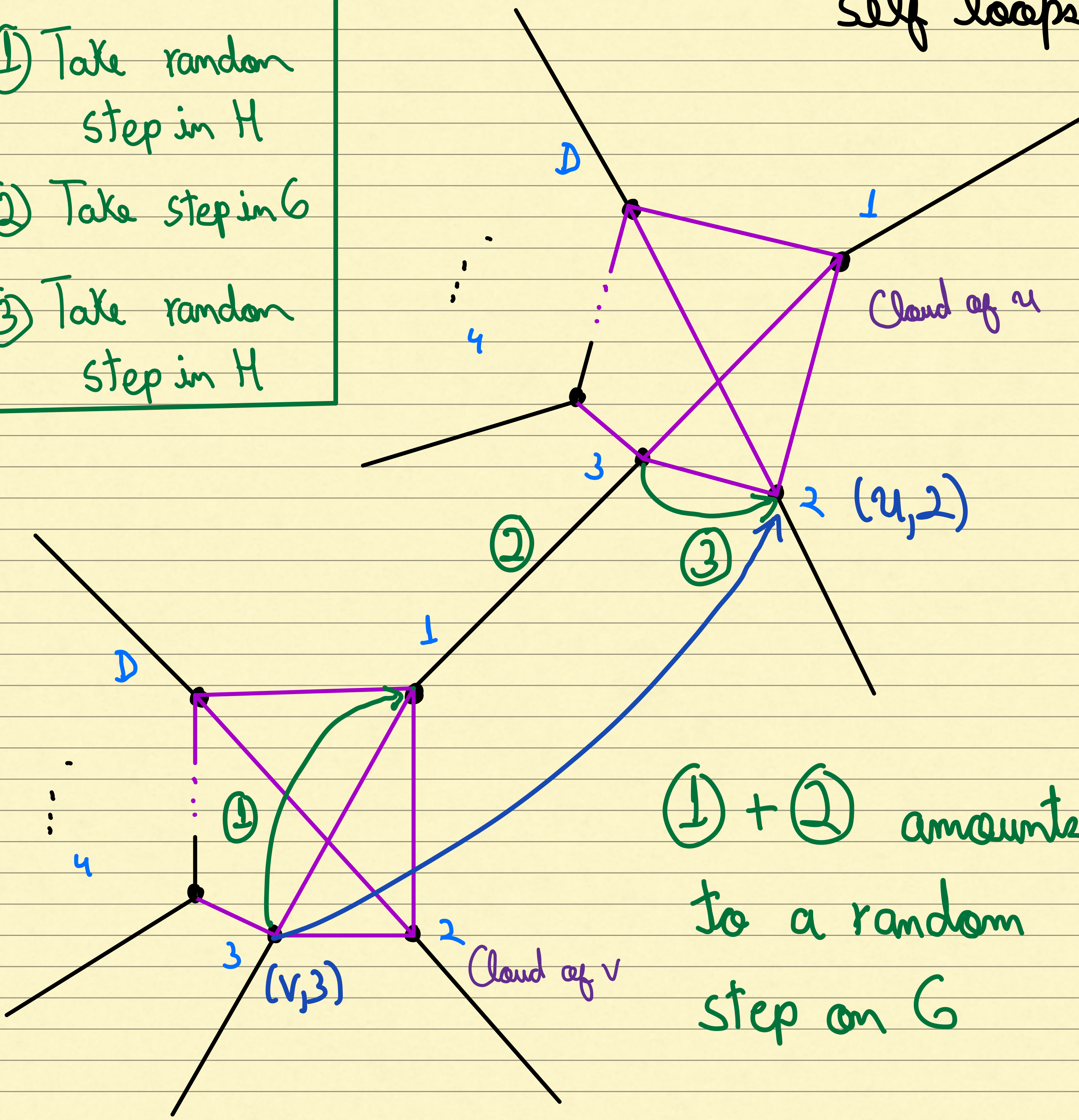
③ Take random step in H



Zig-Zag Product

Special Case: $G \circledZ H$ with H complete graph with self loops

1. Take random step in H
2. Take step in G
3. Take random step in H



1 + 2 amounts to a random step on G

Zig-Zag Product

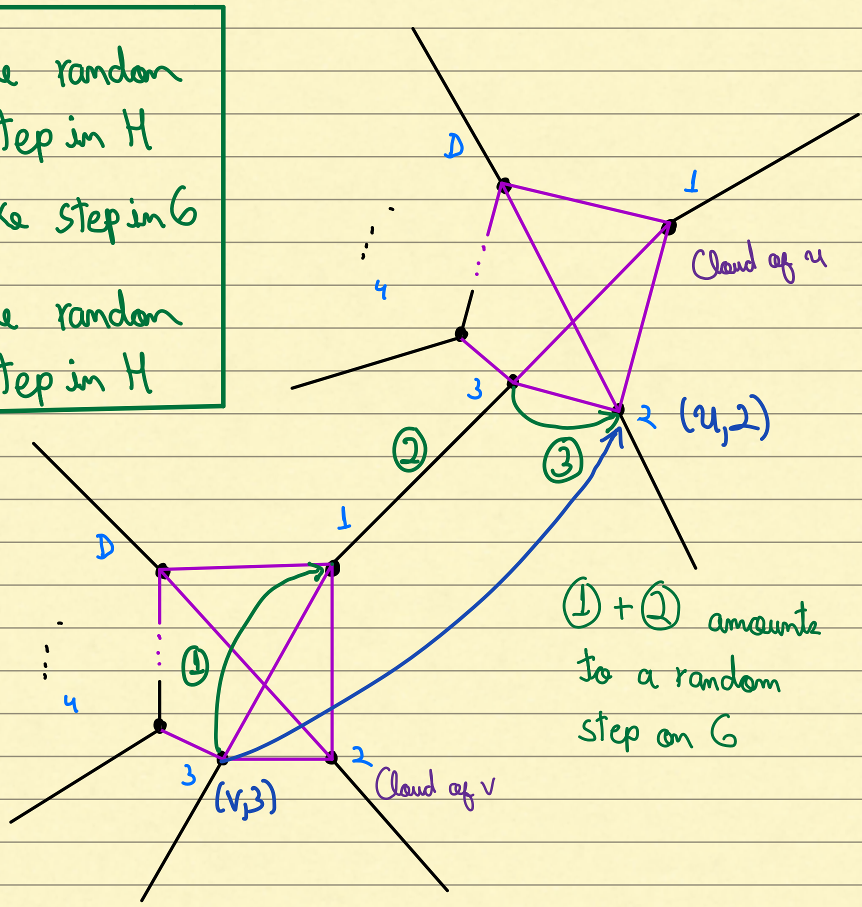
Special Case: $G \textcircled{z} H$ with H complete graph with self loops

$$\left(I \otimes J/_D\right) \tilde{G} \left(I \otimes J/_D\right) = A_G \otimes J/_D$$

① Take random step in H

② Take step in G

③ Take random step in H



Zig-Zag Product

Special Case : $G \textcircled{z} H$ with H complete graph with self loops

Fact: $(I \otimes J_D)\,\tilde{G}\,(I \otimes J/D) = A_G \otimes J/D$

$$\lambda(A_G \otimes J/D) = \lambda(G)$$

Zig-Zag Product

Special Case : $G \textcircled{z} H$ with H complete graph with self loops

$$\boxed{\text{Fact: } \left(I \otimes J/_D\right) \tilde{G} \left(I \otimes J/_D\right) = A_G \otimes J/_D \qquad \lambda\left(A_G \otimes J/_D\right) = \lambda(G)}$$

$$|V(G \textcircled{z} H)| = |V(G)| \cdot |V(H)| \quad ✓$$

$$\lambda(G \textcircled{z} H) = \lambda(G) \quad ✓$$

Degree $G \textcircled{z} H$ is D^2 ✗

Zig-Zag Product

$G \,ⓩ\, H$ with H an expander

Recall that $A_H \approx J/D$

$$\boxed{\begin{array}{l}\text{Fact: } (I \otimes J/D)\,\tilde{G}\,(I \otimes J/D) = A_G \otimes J/D \\ \qquad \lambda(A_G \otimes J/D) = \lambda(G)\end{array}}$$

Zig-Zag Product

$G \textcircled{z} H$ with H an expander

Recall that $A_H \approx J/D$

$$A_H = \frac{J}{D} + \lambda E \qquad \text{with } \|E\|_{op} \leq 1$$

$$(I \otimes A_H)\tilde{G}(I \otimes A_H) =$$

$$\left(I \otimes \frac{J}{D}\right)\tilde{G}\left(I \otimes J/D\right)$$

$$+ \lambda\left(I \otimes J/D\right)\tilde{G}\left(I \otimes E\right)$$

$$+ \lambda\left(I \otimes E\right)\tilde{G}\left(I \otimes J/D\right)$$

$$+ \lambda^2\left(I \otimes E\right)\tilde{G}\left(I \otimes E\right)$$

Zig-Zag Product

$G \textcircled{z} H$ with H an expander

Recall that $A_H \approx J/D$

$$A_H = \frac{J}{D} + \lambda E \qquad \text{with } \|E\|_{op} \leq 1$$

$$(I \otimes A_H)\tilde{G}(I \otimes A_H) =$$

$$\left(I \otimes \frac{J}{D}\right)\tilde{G}\left(I \otimes J/D\right) \rightarrow \lambda_1$$

$$\left.\begin{aligned} &+ \lambda\left(I \otimes J/D\right)\tilde{G}(I \otimes E) \\ &+ \lambda(I \otimes E)\tilde{G}(I \otimes J/D) \end{aligned}\right\} \rightarrow 2\lambda_2$$

$$+ \lambda^2 (I \otimes E)\tilde{G}(I \otimes E) \rightarrow \lambda_2^2$$

$$\boxed{\begin{aligned} &\text{Fact: } \left(I \otimes \tfrac{J}{D}\right)\tilde{G}\left(I \otimes J/D\right) = A_G \otimes J/D \\ &\qquad \lambda(A_G \otimes J/D) = \lambda(G) \end{aligned}}$$

Zig-Zag Product

$G \textcircled{z} H$ is obtained from $G \textcircled{r} H$ by a "zig-zag walk"

Theorem [Reingold-Vadhan-Wigderson'00]

If G is (n, D, λ_1)-graph and H is (D, d, λ_2)-graph, then

$G \textcircled{z} H$ is $(nD, d^2, \lambda_1 + 2\lambda_2 + \lambda_2^2)$-graph

Zig-Zag Product

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[Construct larger and larger expanders starting from a constant size one]

Near-optimal Spectral Expanders

Theorem [Alon-Boppana]

$$\lambda(G) \geq 2\sqrt{d-1} - o_n(1)$$

Ramanujan Graphs

$$\lambda(G) \leq 2\sqrt{d-1}$$

Near-optimal Spectral Expanders

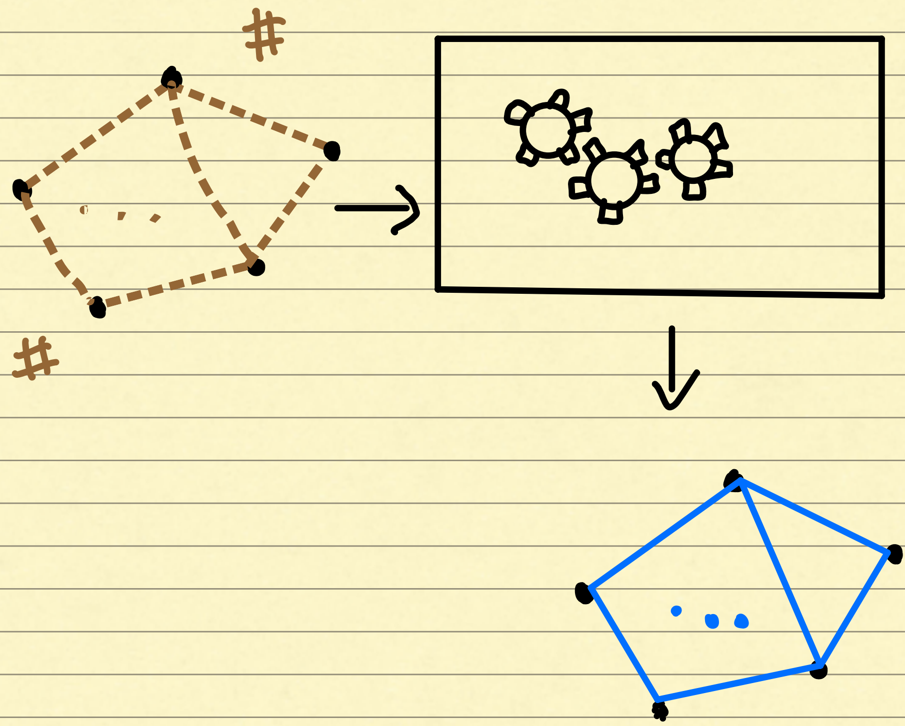
Ramanujan Graphs

$$\lambda(G) \leq 2\sqrt{d-1}$$

What is required to be a near optimal expander?

Near-optimal Spectral Expanders

Theorem [J-Mittal-Roy-Wigderson'22]
Any expander can be transformed into an almost Ramanujan one



Near-optimal Spectral Expanders

Theorem [J-Mittal-Roy-Wigderson'22]
Any expander can be transformed into an almost Ramanujan one

Corollary [JMRW'22]
All "expanding groups" admit almost optimal expanders

Applications to

- "Generic" Expanders
- Cayley Graphs
- Quantum Expanders
- Monotone Expanders
- Dimension Expanders
- Codes

Near-optimal Spectral Expanders

Theorem [J-Mittal-Roy-Wigderson'22]
Any expander can be transformed into an almost Ramanujan one

Key Technique: higher-order zig-zag for operators

(building on the work of Ben-Aroya and Ta-Shma)

Thank You!